Twisted Euler transform of differential equations with an irregular singular point

Kazuki Hiroe*

Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan.

Abstract

In [8], N. Katz introduced the notion of the middle convolution on local systems. This can be seen as a generalization of the Euler transform of Fuchsian differential equations. In this paper, we consider the generalization of the Euler transform, the twisted Euler transform, and apply this to differential equations with irregular singular points. In particular, for differential equations with an irregular singular point of irregular rank 2 at $x = \infty$, we describe explicitly changes of local datum caused by twisted Euler transforms. Also we attach these differential equations to Kac-Moody Lie algebras and show that twisted Euler transforms correspond to the actions of Weyl groups of these Lie algebras.

1 Introduction

For a function $f(x)$, the following integral

$$I_a^{\lambda} f(x) = \frac{1}{\Gamma(\lambda)} \int_a^x (x-t)^{\lambda-1} f(t)\, dt$$

is called the Riemann-Liouville integral for $a, \lambda \in \mathbb{C}$. If we take a function $f(x) = (x-a)^{\alpha}\phi(x)$ where $\alpha \in \mathbb{C}\backslash\mathbb{Z}_{<0}$ and $\phi(x)$ is a holomorphic function on a neighborhood of $x = a$ and $\phi(a) \neq 0$, then it is known that

$$I_a^{-n} f(x) = \frac{d^n}{dx^n} f(x).$$

Hence one can consider the Riemann-Liouville integral to be a fractional or complex powers of derivation $\partial = \frac{d}{dx}$. This may allow us to write $\partial^{\lambda} f(x) = I_a^{-\lambda} f(x)$ formally.

*E-mail:kazuki@ms.u-tokyo.ac.jp

Moreover one can show a generalization of the Leibniz rule,

$$\partial^{\lambda} p(x)\psi(x) = \sum_{i=0}^{n} \binom{\lambda}{i} p^{(i)}(x)\partial^{\lambda-i}\phi(x),$$

if $p(x)$ is a polynomial of degree equal to or less than n.

Now let us consider a differential operator with polynomial coefficients,

$$P(x,\partial) = \sum_{i=0}^{n} a_i(x)\partial^i.$$

The above Leibniz rule assures that

$$\partial^{\lambda+m} P(x,\partial)\partial^{-\lambda}$$

gives the new differential operator with polynomial coefficients if we choose a suitable $m \in \mathbb{Z}$. Moreover if $f(x)$ satisfies $P(x,\partial)f(x) = 0$ and $I_a^{-\lambda} f(x)$ is well-defined for some $a, \lambda \in \mathbb{C}$, then we can see that

$$\begin{aligned}\partial^{\lambda+m} P(x,\partial)\partial^{-\lambda} I_a^{-\lambda} f(x) &= \partial^{\lambda+m} P(x,\partial)\partial^{-\lambda+\lambda} f(x)\\ &= \partial^{-\lambda+m} P(x,\partial) f(x)\\ &= 0.\end{aligned}$$

Hence ∂^{λ} turns a differential equation with polynomial coefficients $P(x,\partial)u = 0$ into a new differential equation with polynomial coefficients $Q(x,\partial)u = 0$, and moreover a solution of $Q(x,\partial)u = 0$ is given by a solution of $P(x,\partial)u = 0$ if the Riemann-Liouville integral is well-defined. This correspondence of differential equations is called the Euler transform.

For example, let us take the differential equation of the Gauss hypergeometric function,

$$x(1-x)\partial^2 u + (\gamma - (\alpha+\beta+1)x)\partial u - \alpha\beta u = 0. \tag{1.1}$$

Then we can see that

$$\begin{aligned}&\partial^{-\beta}(x(1-x)\partial^2 + (\gamma - (\alpha+\beta+1)x)\partial - \alpha\beta)\partial^{\beta-1}\\ &= x(1-x)\partial + ((\gamma-\beta) - (\alpha-\beta+1)x).\end{aligned}$$

And it is not hard to see that the general solution of $x(1-x)\partial + ((\gamma - \beta) - (\alpha-\beta+1)x)u = 0$ is given by constant multiples of $x^{\beta-\gamma}(1-x)^{\alpha-\gamma+1}$. Hence solutions of (1.1) are

$$I_c^{\beta-1} x^{\beta-\gamma}(1-x)^{\alpha-\gamma} = \frac{1}{\Gamma(-\beta)} \int_c^x t^{\beta-\gamma}(1-t)^{\alpha-\gamma}(x-t)^{-\beta}\, dt$$

for $c = 0, 1, \infty$.

This argument tells us that by the Euler transform $\partial^{\beta-1}$, we can reduce (1.1) to an "easier" one,

$$x(1-x)\partial + ((\gamma-\beta) - (\alpha-\beta+1)x).$$

And solutions of (1.1) can be obtained from this "easy" equation.

This can be applicable to differential equations with an irregular singular point. For example, let us consider the differential equations,

$$x\partial^2 u + (\gamma - x)\partial u - \alpha u = 0, \tag{1.2}$$

$$\partial^2 u - x\partial u + \alpha u = 0. \tag{1.3}$$

The first one is the differential equation of the Kummer confluence hypergeometric function. The second one is the one of the Hermite-Weber function. Then we have

$$\begin{aligned}&\partial^{-\alpha}(x\partial^2 + (\gamma - x)\partial - \alpha)\partial^{\alpha-1}\\ &= x\partial + ((\gamma-\alpha) - x),\end{aligned}$$

and

$$\begin{aligned}&\partial^{\alpha}(\partial^2 - x\partial + \alpha)\partial^{-\alpha-1}\\ &= \partial - x.\end{aligned}$$

Solutions of these differential equations are $x^{\alpha-\gamma}e^x$ and $e^{\frac{x^2}{2}}$ up to constant multiples. Hence solutions of (1.2) and (1.3) are given by

$$\frac{1}{\Gamma(-\alpha)}\int_c^x (x-t)^{-\alpha} t^{\alpha-\gamma} e^t \, dt$$

for $c = 0, \infty$, and

$$\begin{aligned}\frac{1}{\Gamma(\alpha)}\int_{-\infty}^x (x-t)^{\alpha} e^{\frac{t^2}{2}} \, dt &= \frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha} e^{\frac{(x-t)^2}{2}} \, dt\\ &= \frac{e^{\frac{x^2}{2}}}{\Gamma(\alpha)}\int_0^{\infty} e^{\frac{t^2}{2}-tx} t^{\alpha} \, dt.\end{aligned}$$

These facts tell us that the Euler transform is a good tool to study a solution of a differential equation from a solution of an easier differential equation. Then a question arises.

Can one always reduce differential equations to "easy" ones by Euler transforms? If not, find a good class of differential equations which can be reduced to easy one.

An answer of this question is known for Fuchsian differential equations. Let us consider a system of Fuchsian differential equations of the form,

$$\frac{d}{dx}Y(x) = \sum_{i=1}^{r} \frac{A_i}{(x-c_i)}Y(x), \tag{1.4}$$

where A_i are $n \times n$ complex matrices, $Y(x)$ is a $\mathbb{C}^n$-valued function and $c_i \in \mathbb{C}$. In this case, it is known that the Euler transform corresponds to the additive middle convolution (see [5] and [6]). The answer of our question is given by the following theorem.

Theorem 1.1 (Katz [8], Dettweiler-Reiter [5] [6]). *The tuple of $n \times n$ of matrices, $A_1, \ldots, A_r$ is irreducible and linearly rigid, the differential equation (1.4) can be reduced to*

$$\frac{d}{dx}y(x) = \sum_{i=1}^{r} \frac{a_i}{x-c_i}y(x)$$

where $a_i \in \mathbb{C}$ by finite iterations of middle convolutions and additions.

Definitions of terminologies in this theorem can be found in the original papers [5] and [6].

Our purpose of this paper is to extend this theorem to non-Fuchian differential equations.

In this paper, we consider differential equations with polynomial coefficients which have an irregular singular point of rank at most 2 at $x = \infty$ and some regular singular points in $\mathbb{C}$. And we give a generalization of Theorem 1.1.

The organization of this paper is the following. In Section 2, we give a review of some operations on the Weyl algebra. These operations are provided by T. Oshima in [11] to define the Euler transform in the strict way as an operation on the Weyl algebra. We use these operations to define a generalization of the Euler transform, the twisted Euler transform.

Because we treat differential equations with an irregular singular point, solutions of them have asymptotic expansions as formal power series. Hence in Section 3, we study the formal solutions of differential equations. We introduce the notion of semi-simple characteristic exponents here.

Section 4 is one of the main parts of this paper. We focus on differential equations which have an irregular singular point of rank 2 at infinity and regular singular points in $\mathbb{C}$. We also assume that at the irregular singular point, every formal solutions are "normal type", i.e., formal solutions can be written by $e^{p(x)}x^{-\mu}\sum_{s=0}^{\infty} c_s x^{-s}$ for polynomials $p(x)$. Equivalently to say, we assume that differential equations are unramified according to the terminology of the differential Galois theory. Then in Section 4, we investigate the way how the twisted Euler transform changes differential equations, in

other words, how characteristic exponents at singular points are changed by the twisted Euler transform.

In Section 5, we see the relation between differential equations and Kac-Moody Lie algebras. In [4], W. Crawley-Boevey find a correspondence of the systems of Fuchsian differential equations as (1.4) and representations of quivers and moreover Kac-Moody Lie algebras associated with these representations. And he solved the existence problem of systems of differential equations, so-called Deligne-Simpson problem by using the theory of representations of quivers. An analogous work is done by P. Boalch in [3]. Boalch generalizes the result of Crawley-Boevey for the cases which allow an irregular singular point of rank 2. And he also solved Deligne-Simpson problem with an irregular singular point of rank 2.

As an analogue of their works, we attach a Kac-Moody Lie algebra to the concerning differential equation. Moreover we show that twisted Euler transforms correspond to simple reflections on the Cartan subalgebra of this Lie algebra. By using this correspondence, we prove the following main theorem.

Theorem 1.2. *Let us take $P(x,\partial) \in W[x,\xi]$ as in Definition 4.12. If $\operatorname{idx} P > 0$, then $P(x,\partial)$ can be reduced to*

$$(\partial - \alpha x - \beta)^n$$

for some $\alpha, \beta \in \mathbb{C}$ and $n \in \mathbb{Z}_{>0}$ by finite iterations of twisted Euler transforms and additions at regular singular points.

Notations used in this theorem are explained in the subsequent sections.

In Section 6, we consider the confluence of Fuchsian differential equations. It is known that differential equations of Kummer confluent hypergeometric function and Hermite weber functions are obtained from the differential equation of the Gauss hypergeometric function by the limit transitions. In Section 6, as a generalization of this fact, we show the following.

Theorem 1.3. *Take $P(x,\partial) \in W[x,\xi]$ as in Definition 4.12. If $\operatorname{idx} P > 0$, then $P(x,\partial)$ can be obtained by the limit transition of a Fuchsian $Q(x,\partial) \in W[x,\xi]$ of $\operatorname{idx} Q = \operatorname{idx} P$.*

Meanwhile, in Appendix, we consider the differential equation which has regular singular point at $x = \infty$ and arbitrary singularities at any other points. And we give a necessary and sufficient condition to reduce the order of this differential equation by Euler transform $E(0,\mu)$.

Theorem 1.4. *Let us take $P(x,\partial) \in W[x]$ which has regular singular point at $x = \infty$ and semi-simple exponents*

$$\{[\mu_1]_{n_1}, \ldots, [\mu_l]_{n_l}\},$$

where $\sum_{i=1}^{l} n_i = n = \operatorname{ord} P$, $\mu_i \notin \mathbb{Z}$ *and* $\mu_i - \mu_j \notin \mathbb{Z}$ *if* $i \neq j$. *Then we have*

$$\operatorname{ord} E(0, \mu_i - 1)P(x, \partial) < \operatorname{ord} P$$

if and only if

$$\deg P - \operatorname{ord} P < n_i.$$

There are many other works about middle convolutions, equivalently to say, Euler transforms of differential equations with irregular singular points. T. Kawakami considers generalization of middle convolutions to systems of differential equations with irregular singular point which are called generalized Okubo systems in [9]. K. Takemura [13] and D. Yamakawa [15] consider the middle convolutions for the system of the form

$$\frac{d}{dx}Y(x) = \sum_{i=1}^{r} \sum_{j=1}^{k_i} \frac{A_{ij}}{(x - c_i)^j} Y(x)$$

where A_{ij} are $n \times n$ complex matrices. In particular, Yamakawa discusses the reduction of the rank of a differential equation which is similar problem to ours.

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2 Operations on localized Weyl algebra with parameters.

In this section, some operations on localized Weyl algebra are introduced. In [11], T. Oshima uses these operators to understand the Euler transform and he constructs a theory of the Euler transform which corresponds to the theory of additive middle convolutions for Schleginger type systems of differential equations studied by Dettweiler and Reiter in [5] and [6].

The Weyl algebra $W[x]$ is the $\mathbb{C}$-algebra generated by x and $\partial = \frac{d}{dx}$ with the relation,

$$[\partial, x] = \partial x - x\partial = 1.$$

The localization of the Weyl algebra is $W(x) = \mathbb{C}(x) \otimes_{\mathbb{C}} W[x]$ where $\mathbb{C}(x)$ is the quotient field of $\mathbb{C}[x]$ the polynomial ring with complex coefficients. For indeterminants $\xi = (\xi_1, \ldots, \xi_n)$, we consider $\mathbb{C}(\xi)$ the field of rational

functions with complex coefficients and fix an algebraic closure $\overline{\mathbb{C}(\xi)}$ of $\mathbb{C}(\xi)$. Then we can define the Weyl algebra with parameters

$$W[x,\xi] = \overline{\mathbb{C}(\xi)} \otimes_{\mathbb{C}} W[x].$$

We can also define the localized Weyl algebra with parameters $W(x,\xi) = \overline{C(\xi)} \otimes_{\mathbb{C}} W(x)$.

The element $P \in W(x,\xi)$ (resp. $P \in W[x,\xi]$) is uniquely written by

$$P = \sum_{i=0}^{m} p_i(x;\xi)\partial^i$$

with $p_i(x,\xi) \in \mathbb{C}(x,\xi) = \overline{\mathbb{C}(\xi)} \otimes_{\mathbb{C}} \mathbb{C}(x)$ (resp. $p_i(x,\xi) \in \mathbb{C}[x,\xi] = \overline{\mathbb{C}(\xi)} \otimes_{\mathbb{C}} \mathbb{C}[x]$) for $i = 0,\dots,m$. According to this representation, the order of P denoted by $\operatorname{ord} P$ is defined by the maximum integer i such that $p_i(x,\xi) \neq 0$. We also define the degree of $P \in W[x;\xi]$ by maximum of degrees of $p_i(x;\xi)$ as polynomials of x, and denoted by $\deg P$.

Definition 2.1. *For $P(x,\partial) \in W(x,\xi)$ of $\operatorname{ord} P > 0$, we say that $P(x,\partial)$ is irreducible when it is satisfied that if we can write $P = QR$ for some $Q, R \in W(x,\xi)$ then $\operatorname{ord} Q \cdot \operatorname{ord} R = 0$.*

We recall some operations on $W[x,\xi]$ and $W(x,\xi)$. Details of these operations can be found in [11].

Definition 2.2 (The Fourier-Laplace transform)**.** *The Fourier-Laplace transform on $W[x,\xi]$ is the following algebra isomorphism*

$$\begin{array}{rccll} \mathcal{L}\colon & W[x,\xi] & \longrightarrow & W[x,\xi] & \\ & x & \longmapsto & -\partial & \\ & \partial & \longmapsto & x & \\ & \xi_i & \longmapsto & \xi_i & (i = 1,\dots,n). \end{array}$$

Sometimes we identify $P \in W(x,\xi)$ and $f(x)P$ for $f(x) \in \mathbb{C}(x,\xi)$ because differential equations $P(x,\partial)u = 0$ and $f(x)P(x,\partial)u = 0$ can be identified. If for $P, Q \in W(x,\xi)$, there exist $f(x) \in \mathbb{C}(x,\xi)$ such that $P = f(x)Q$, then we write

$$P \sim Q.$$

Definition 2.3 (The reduced representative)**.** *The reduced representative of a nonzero element $P \in W[x,\xi]$ is an element of $\mathbb{C}(x,\xi)P \cap W[x,\xi]$ of the minimal degree and denoted by $\mathrm{R}P$.*

Definition 2.4. *For $h \in \mathbb{C}(x,\xi)$, we define the automorphism of $W(x,\xi)$ by*

$$\begin{array}{rccll} \operatorname{Adei}(h)\colon & W(x;\xi) & \longrightarrow & W(x;\xi) & \\ & x & \longmapsto & x & \\ & \partial & \longmapsto & \partial - h & \\ & \xi_i & \longmapsto & \xi_i & (i = 1,\dots,n). \end{array}$$

Definition 2.5 (The addition at the singular point)**.** *For $c \in \mathbb{C}$ and $f(\xi) \in \mathbb{C}(\xi)$ we call the operator* $\mathrm{Adei}(\frac{f(\xi)}{x-c})$ *the addition at the singular point c and denote it by* $\mathrm{Ad}((x-c)^{f(\xi)})$. *We also define* $\mathrm{RAd}((x-c)^{f(\xi)}) = \mathrm{R} \circ \mathrm{Ad}((x-c)^{f(\xi)})$.

Remark 2.6. *Let $f(x)$ and $g(x)$ be sufficiently many differentiable functions. Then the Leibniz rule tells us that*

$$\frac{d}{dx} f(x)g(x) = (f(x)\frac{d}{dx} + \frac{d}{dx} f(x))g(x).$$

Hence for some $c, \lambda \in \mathbb{C}$ we have

$$(x-c)^{\lambda} \frac{d}{dx}(x-c)^{-\lambda} g(x) = (\frac{d}{dx} - \frac{\lambda}{x-c})g(x).$$

Therefore $\mathrm{Adei}(\frac{\lambda}{x-c})\frac{d}{dx}$ *corresponds to*

$$(x-c)^{\lambda} \frac{d}{dx}(x-c)^{-\lambda}.$$

Definition 2.7 (The $e^{p(x)}$-twisting)**.** *For $p(x) \in \mathbb{C}[x]$, we define*

$$\mathrm{Ade}(p(x)) = \mathrm{Adei}(p'(x))$$

and call this the $e^{p(x)}$-twisting. Here $p'(x) = \frac{d}{dx}p(x)$.

Remark 2.8. *As well as additions at singular points, the $e^{p(x)}$-twisting corresponds to the operation*

$$\frac{d}{dx} \longmapsto e^{p(x)} \frac{d}{dx} e^{-p(x)} = (\frac{d}{dx} - p'(x)).$$

Definition 2.9 (The twisted Euler transform)**.** *For $\alpha \in \mathbb{C}$ and $f(\xi) \in \mathbb{C}(\xi)$ we define the operator on $W[x, \xi]$,*

$$E(\alpha, f(\xi)) = \mathcal{L} \circ \mathrm{RAd}((x+\alpha)^{-f(\xi)}) \circ \mathcal{L}^{-1} \circ \mathrm{R}.$$

For $\overline{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{C}^m$ and $F(\xi) = (f_1(\xi), \ldots, f_m(\xi)) \in (\mathbb{C}(\xi))^m$, we define the operator

$$E(p(x); \overline{\alpha}, F(\xi)) = \mathrm{Ade}(p(x)) \circ \prod_{i=1}^{m} E(\alpha_i, f_i(\xi)) \circ \mathrm{Ade}(-p(x)).$$

We call $E(\alpha, f(\xi))$ and $\mathrm{E}(p(x); \overline{\alpha}, F(\xi))$ twisted Euler transforms. In particular we call $\mathrm{E}(0, f(\xi))$ an Euler transform.

Remark 2.10. *A twisted Euler transform $E(\alpha, f(\xi))$ corresponds to the following integral transform*

$$\int_{-i\infty}^{i\infty}(y+\alpha)^{-f(\xi)}\int_{c}^{\infty} g(x)e^{-xy}\,dx\,e^{zy}\,dy \\ = \frac{1}{\Gamma(f(\xi))}\int_{c}^{z} g(x)(z-x)^{f(\xi)-1}e^{-\alpha(z-x)}\,dx.$$

If we put $\alpha = 0$, then this is a Riemann-Liouville integral. Hence $E(0, f(\xi))$ can be seen as the classical Euler transform.

Remark 2.11. *A twisted Euler transform can be written as the composition of $e^{p(x)}$-twistings and an Euler transform. We define the parallel displacement for $\alpha \in \mathbb{C}$ by*

$$\begin{array}{rccccl} \mathrm{P}(h)\colon & W(x;\xi) & \longrightarrow & W(x;\xi) & \\ & x & \longmapsto & x-\alpha & \\ & \partial & \longmapsto & \partial & \\ & \xi_i & \longmapsto & \xi_i & \quad (i=1,\dots,n). \end{array}$$

Then we can see that $\mathcal{L}^{-1}\mathrm{Ade}(\alpha x) = \mathrm{P}(-\alpha)\mathcal{L}^{-1}$ and $\mathrm{Ade}(-\alpha x)\mathcal{L} = \mathcal{L}\mathrm{R}(-\alpha)$. Hence we have

$$E(\alpha, f(\xi)) = \mathrm{Ade}(\alpha x)E(0, f(\xi))\mathrm{Ade}(-\alpha x).$$

Proposition 2.12. *We take an element $P(x,\partial) \in W(x,\xi)$ and assume that $P(x,\partial)$ is irreducible and $\operatorname{ord} P > 1$. Then we have followings.*

1. *We have*
$$E(*,0)P \sim P$$
where we can put any $\alpha \in \mathbb{C}$ into $*$.

2. *If $E(\alpha, f(\xi))P$ is irreducible and $\operatorname{ord} E(\alpha, f(\xi))P > 1$ for an $\alpha \in \mathbb{C}$ and an $f(\xi) \in \mathbb{C}(\xi)$, we have*
$$E(\alpha, -f(\xi))E(\alpha, f(\xi))P = E(\alpha, 0)P \sim P$$
.

Before proving this proposition, we see the following lemma.

Lemma 2.13. *Suppose that $P \in W[x,\xi]$ is irreducible. If $E(*,0)P = g(x) \in \mathbb{C}[x,\xi]$, then there exists $f(x) \in \mathbb{C}[x,\xi]$ such that*

$$P \sim f(\partial)g(x)$$

and the degree of $f(x)$ as the polynomial of x is at most one.

Proof. By the assumption, we have

$$E(*,0)P = \mathcal{L}\mathrm{R}\mathcal{L}^{-1}\mathrm{R}P = g(x).$$

This implies that

$$g(\partial) = \mathcal{L}^{-1}g(x) = \mathrm{R}\mathcal{L}^{-1}\mathrm{R}P,$$

that is, there exists an $f(x) \in \mathbb{C}[x,\xi]$ such that

$$\mathcal{L}^{-1}\mathrm{R}P = f(x)g(\partial).$$

Equivalently we have

$$P \sim f(-\partial)g(x).$$

By the irrducibility of P, $f(x)$ must be an irreducible polynomial, i.e., its degree is at most one. $\square$

proof of Proposition 2.12. $\square$

Let us put $Q = E(*,0)P = \mathcal{L}\mathrm{R}\mathcal{L}^{-1}\mathrm{R}P$. Then we have

$$\mathrm{R}\mathcal{L}^{-1}\mathrm{R}P = \mathcal{L}^{-1}Q.$$

Hence there exists an $f(x) \in \mathbb{C}[x,\xi]$ such that

$$\mathcal{L}^{-1}\mathrm{R}P = f(x)\mathcal{L}^{-1}Q,$$

that is, we have

$$P \sim f(-\partial)Q.$$

By Lemma 2.13 and the assumption $\operatorname{ord} P > 1$, we can see that $\operatorname{ord} Q \neq 0$. Since P is irreducible, we have $f(\partial) \in \mathbb{C}$. Hence we obtain

$$E(*;0)P \sim P.$$

Let us recall that

$$\mathrm{RAd}((x+\alpha)^{f(\xi)}) \circ \mathrm{RAd}((x+\alpha)^{-f(\xi)})Q \sim Q$$

for $Q \in W[x,\xi]$.

By the assumption, $E(\alpha, f(\xi))P$ is irreducible and $\operatorname{ord} E(\alpha, f(\xi))P > 1$. Hence we obtain

$$\begin{aligned}
&E(\alpha,-f(\xi))E(\alpha,f(\xi)) = E(\alpha,-f(\xi))E(*,0)E(\alpha,f(\xi))P \\
&= \mathcal{L}^{-1}\mathrm{RAd}((x+\alpha)^{f(\xi)})\mathrm{RAd}((x+\alpha)^{-f(\xi)})\mathcal{L} \circ \mathrm{R}P \\
&\sim P.
\end{aligned}$$

3 Formal solutions of differential equations

We study formal solutions of differential equations with Laurent series coefficients in this section. Although this section will contain many well-known facts, we will give proofs of these for the completeness of the paper. One of the purpose of this section is to understand when a system of fundamental solutions has no logarithmic singularities even though differences of characteristic exponents are integers. One of the answer of this question is given by Oshima for Fuchsian differential equations in [11]. We apply his result to formal solutions of differential equations with Laurent series coefficients.

3.1 Formal solutions at infinity.

Let $\mathcal{K}$ be the field of Laurent series of x^{-1}, also write $\mathbb{C}[[x^{-1}]][x]$, and $\mathcal{K}\langle\partial\rangle$ the ring of differential operators with coefficients in $\mathcal{K}$.

Let us consider an element in $\mathcal{K}\langle\partial\rangle$,

$$P(x,\partial) = a_n(x)\partial^n + a_{n-1}(x)\partial^{n-1} + \ldots + a_1(x)\partial + a_0(x)$$

for $a_i(x) \in \mathbb{C}[[x^{-1}]][x]$. For $a(x) = \sum_{n=-\infty}^{\infty} a_n x^{-n} \in \mathbb{C}[[x^{-1}]][x]$, $v(a(x))$ denote the valuation of $a(x)$, i.e., $v(a(x)) = \min\{s \in \mathbb{Z} \mid a_s \neq 0\}$. We consider "normal" formal solutions near $x = \infty$, i.e., solutions of the form $x^\mu f(x)$ for some $\mu \in \mathbb{C}$ and $f(x) \in \mathbb{C}[[x^{-1}]]$ ($f(\infty) \neq 0$).

The group ring generated by $\{x^\mu \mid \mu \in \mathbb{C}\}$ is denoted by $\mathbb{C}[x^\mu]$. Also the polynomial ring of $\log x^{-1}$ with coefficients in $\mathbb{C}[[x^{-1}]]$ is denoted by $\mathbb{C}[[x^{-1}]][\log x^{-1}]$. We consider the tensor product $\mathbb{C}[x^\mu] \otimes_{\mathbb{C}} \mathbb{C}[[x^{-1}]][\log x^{-1}]$ as $\mathbb{C}$-algebras and we simply write these elements $x^\mu \otimes h(x)$ by $x^\mu h(x)$ for $\mu \in \mathbb{C}$ and $h(x) \in \mathbb{C}[[x^{-1}]][\log x^{-1}]$.

As an analogue of systems of fundamental solutions around regular singular points, we consider subspaces of $\mathbb{C}[x^\mu] \otimes_{\mathbb{C}} \mathbb{C}[[x^{-1}]][\log x^{-1}]$ which are spanned by formal power series with logarithmic terms.

Definition 3.1. *Let us take $\mu_1, \ldots \mu_r \in \mathbb{C}$, such that $\mu_i - \mu_j \notin \mathbb{Z}$ for $i \neq j$, increasing sequences of positive integers $0 = n_0^i < \cdots < n_{l_i}^i$ for $i = 1, \ldots, r$ and $m_j^i \in \mathbb{Z}_{\geq 0}$ for $i = 1, \ldots, r$, $j = 0, \ldots, l_i$. We put $h_j^i = \sum_{k=1}^{j} m_k^i$ and $n = \sum_{i=1}^{r} h_{l_i}^i$.*

Then we consider the following n functions,

$$f(\mu_i + n_j^i; k)(x) = x^{\mu_i + n_j^i} \sum_{l=0}^{h_j^i + k} \binom{h_j^i + k}{l} c^{ij}_{h_j^i + k - l}(x)(\log x^{-1})^l,$$

where $\binom{k}{l}$ are binomial coefficients, $c_k^{ij}(x) \in \mathbb{C}[[x^{-1}]]$ and $c_0^{i0}(\infty) \neq 0$. We call the $\mathbb{C}$-vector subspace spanned by these functions the space of formal regular series.

Lemma 3.2. *We use the same notations as in Definition 3.1. Let V be a n-dimensional space of formal regular series. For $i = 1, \ldots, r$, we define $\mathbb{C}$-linear maps*

$$\begin{array}{rccc} \Phi_i\colon & V & \longrightarrow & \mathbb{C}[x^{\mu}] \otimes_{\mathbb{C}} \mathbb{C}[[x^{-1}]][\log x^{-1}] \\ & f(x) & \mapsto & \partial \frac{f(x)}{f(\mu_i;0)(x)}. \end{array}$$

Then each image $\Phi_i(V)$ is $n-1$-dimensional space of formal regular series.

Proof. We can see that

$$\partial f(\mu_i + n^i_j; k)(x) = x^{\mu_i + n^i_j - 1} \sum_{l=0}^{h^i_j + k} \binom{h^i_j + k}{l} H^{ij}_{h^i_j + k - l}(x)(\log x^{-1})^l,$$

where

$$H^{ij}_{h^i_j+k-l} = \{((\mu_i + n_{k+1}) + x\partial)c^{ij}_{h^i_j+k-l}(x) - (h^i_j + k - l)c^{ij}_{h^i_j+k-l-1}(x)\}.$$

Also we can see that

$$\partial \frac{1}{f(\mu_{i'};0)(x)} = x^{-\mu_{i'}-1} F^{i'}(x),$$

where $F^{i'}(x) = ((-\mu_{i'} + x\partial)c^{i'0}_0(x)^{-1})$. Clearly $F^{i'}(\infty) \neq 0$.

Hence we have

$$\partial(\frac{f(\mu_i + n^i_j;k)(x)}{f(\mu_{i'};k)(x)}) = x^{\mu_i - \mu_{i'} + n^i_j - 1} \sum_{l=0}^{h^i_j+k} \binom{h^i_j + k}{l} G^{ij}_{h^i_j+k-l}(x)(\log x^{-1})^l.$$

where

$$G^{ij}_k(x) = H^{ij}_k(x) f(\mu_{i'};0)(x)^{-1} + c^{ij}_k(x) F^{i'}(x).$$

The image $\Phi_{i'}(V)$ is spanned by $\left\{\partial(\frac{f(\mu_i+n^i_j;k)(x)}{f(\mu_{i'};k)(x)})\right\}_{i,j,k}$. Since $G^{i0}_0(x) = ((\mu_i - \mu_{i'}) + x\partial)c^{i0}_0(x)c^{i'0}_0(x)^{-1}$, we have $G^{i0}_0(\infty) \neq 0$ if $i \neq i'$. If $i = i'$, $G^{i0}_0(x) = 0$. However, we have

$$\partial(\frac{f(\mu_{i'};1)(x)}{f(\mu_{i'};0)(x)}) = \partial(c^{i'0}_1(x)c^{i'0}_0(x)^{-1} + \log x^{-1}).$$

Hence we obtain that $G^{i0}_1(x) = (1 + x\partial c^{i'0}_1(x)c^{i'0}_0(x)^{-1})$ and $G^{i0}_1(\infty) \neq 0$. Hence $\Phi_{i'}(V)$ is the space of formal regular series.

□

Lemma 3.3. *Let $P(x, \partial) \in \mathcal{K}\langle\partial\rangle$ be*

$$P(x, \partial) = a_n(x)\partial^n + a_{n-1}(x)\partial^{n-1} + \ldots + a_1(x)\partial + a_0(x),$$

for $a_i(x) \in \mathbb{C}[[x^{-1}]][x]$. It can be written by

$$P(x, \partial) = \sum_{s}^{\infty} x^{\rho-s} P_s(\partial),$$

where $\rho = \max\{v(a_i(x)) \mid i = 0, \ldots, n\}$ and $P_s(x) \in \mathbb{C}[x]$. Then the followings are equivalent

1. *$P_i^{(j)}(0) = \partial^j P_i(x)|_{x=0} = 0$ for $i + j < l$.*
2. *$v(a_{l-i-1}(x)) < \rho - i$ $(i = 0, \ldots, l-1)$.*

Proof. The condition 1 is equivalent to that $P(x, \partial)$ is written by

$$P(x, \partial) = x^\rho \partial^l P'_0(\partial) + x^{\rho-1}\partial^{l-1}P'_1(\partial) + \cdots + x^{\rho-l+1}\partial P'_{l-1}(\partial) + x^{\rho-l}P_l(\partial) + \cdots .$$

Then the assertion is obvious. □

Proposition 3.4. *Let $P(x, \partial) \in \mathcal{K}\langle\partial\rangle$ be*

$$P(x, \partial) = a_n(x)\partial^n + a_{n-1}(x)\partial^{n-1} + \ldots + a_1(x)\partial + a_0(x),$$

for $a_i(x) \in \mathbb{C}[[x^{-1}]][x]$. Also we write

$$P(x, \partial) = \sum_{s=0}^{\infty} x^{\rho-s} P_s(\partial)$$

where $P_s(x) \in \mathbb{C}[x]$ and $\deg P_s \leq n$ for $s = 0, 1, 2, \ldots$.

For $l \leq n$, we assume that there exist an l-dimensional space of formal regular series V whose elements are solutions of the differential equation $P(x, \partial)u = 0$.

Then we have

$$P_i^{(j)}(0) = 0 \text{ for } i + j < l,$$

equivalently

$$v(a_{l-i-1}(x)) < \rho - i \ (i = 0, \ldots, l-1).$$

Proof. We prove this by induction on l. We assume that $P(x, \partial)$ has a formal solution $x^\mu h(x)$ for $\nu \in \mathbb{C}$ and $h(x) \in \mathbb{C}[[x^{-1}]]$ $(h(\infty) \neq 0)$. Then the coefficient of x^μ of $P(x, \partial)x^\mu h(x)$, is $P_0(0)h(\infty)$. Hence $P_0(0) = 0$.

We assume that there exists a k-dimensional space V of formal regular series whose elements are solutions of $P(x, \partial)u = 0$. We take an element of V without log term, i.e., an element $\phi(x) = x^\mu h(x)$ for $\mu \in \mathbb{C}$ and $h(x) \in \mathbb{C}[[x^{-1}]]$ $(h(\infty) \neq 0)$. We consider a differential operetor $\tilde{P}(x, \partial) =$

$\phi(x)^{-1}P(x,\partial)\phi(x)$. Then there exits $Q(x,\partial) \in \mathcal{K}\langle\partial\rangle$ such that $\tilde{P}(x,\partial) = Q(x,\partial)\partial$ because $1 = \phi(x)^{-1}\phi(x)$ is a solution of $\tilde{P}(x,\partial)u = 0$. Hence if we consider a $\mathbb{C}$-linear map

$$\begin{array}{cccc} \Phi\colon & V & \longrightarrow & \mathbb{C}[x^{\mu}] \otimes_{\mathbb{C}} \mathbb{C}[[x^{-1}]][\log x^{-1}] \\ & f(x) & \mapsto & \partial \frac{f(x)}{\phi(x)}, \end{array}$$

elements in the image $\Phi(V)$ are formal solutions of $Q(x,\partial)$. By Lemma 3.2, $\Phi(V)$ is a $k-1$-dimensional space of formal regular series and elements of $\Phi(V)$ are formal solutions of $Q(x,\partial)u = 0$ by the construction. Hence by the hypothesis of the induction we have

$$Q_i^j(0) = 0 \text{ for } i + j < k - 1.$$

If we write

$$Q(x,\partial) = \sum_{i=0}^{n-1} q_i(x)\partial^i$$

for $q_k(x) \in \mathbb{C}[[x^{-1}]][x]$ $(k = 0,\dots,n-1)$, the above condition is equivalent to

$$v(q_{k-i-2}(x)) < \rho - i \ (i = 0,\dots,k-2) \tag{3.1}$$

by Lemma 3.3. Also we can write

$$P(x,\partial) = \sum_{i=0}^{n} p_i(x)\partial^i$$

for $p_i(x) \in \mathbb{C}[[x^{-1}]][x]$ $(i = 0,\dots,n)$. If we recall the Leibniz rule,

$$\partial^i f(x)g(x) = \sum_{j=0}^{i} \binom{i}{j} f^{(j)}(x)\partial^{i-j}g(x),$$

the equation $\phi(x)P(x,\partial)\phi(x)^{-1} = Q(x,\partial)\partial$ says that

$$p_{n-i}(x) = \sum_{j=0}^{i} \binom{n-j}{i-j} \frac{\phi^{(i-j)}(x)}{\phi(x)} q_{n-j-1}(x) \ (i = 0,\dots,n).$$

Here we put $q_{-1}(x) = 0$. Since $v(\frac{\phi^{(i)}}{\phi(x)}) \leq -i$, valuations of $p_i(x)$ are bounded as follows,

$$v(p_i(x)) \leq \max\{v(q_{j-1}) - (j-i) \mid j = i,\dots,n\}.$$

If we notice that

$$v(p_{i+1}(x)) \leq \max\{v(q_{j-1}) - (j-i) + 1 \mid j = i+1,\dots,n\},$$

then we have

$$v(p_i(x)) \le \max\{v(p_{i+1}(x)) - 1, v(q_{i-1}(x))\} \ (i = 1, \dots, n).$$

Then above inequalities and (3.1) implies that

$$v(p_{k-i-1}(x)) < \rho - i \ (i = 0, \dots, k-1). \tag{3.2}$$

□

Definition 3.5 (The characteristic equation)**.** *Let us take an element of* $\mathcal{K}\langle\partial\rangle$,

$$P(x, \partial) = \sum_{s=0}^{\infty} x^{\rho - s} P_s(\partial)$$

where $P_s(x) \in \mathbb{C}[x]$ *and* $\deg P_s \le n$ *for* $s = 0, 1, 2, \dots$. *We consider following polynomials of* λ,

$$p_k(\lambda, x) = \sum_{i=0}^{k} (\lambda + i + 1)_{k-i} \frac{1}{(k-i)!} P_i^{(k-i)}(x)$$

for $k = 1, \dots, n$. *Here we put* $(\lambda)_l = \lambda(\lambda+1)\cdots(\lambda+(l-1))$ *for* $l = 1, 2, \dots$ *and* $(\lambda)_0 = 1$. *If* $P_0^{(k)}(0) \ne 0$, *then* $p_k(\lambda, 0)$ *is a polynomial of* λ *of degree* k. *Hence if* $P_0^{(m)}(0) \ne 0$ *and* $P_j^{(i)}(0) = 0$ *for* $i + j < m$, *we call*

$$p_m(\lambda - m, 0) = 0$$

the characteristic equation of $P(x, \partial)$.

Lemma 3.6. *Let us take* $P(x, \partial)$ *as in Definition 3.5. Suppose that* $P_i^{(j)}(0) = 0$ *for* $i + j < m$ *and* $P_0^{(m)}(0) \ne 0$ *for an* $m \le n$.

Also we assume that there exist $\mu_1, \dots \mu_r \in \mathbb{C}$, *such that* $\mu_i - \mu_j \notin \mathbb{Z}$ *for* $i \ne j$, *increasing sequences of positive integers* $0 = n_0^i < \cdots < n_{l_i}^i$ *for* $i = 1, \dots, r$ *and* $m_j^i \in \mathbb{Z}_{>0}$ *for* $i = 1, \dots, r$, $j = 0, \dots, l_i$ *such that* $m = \sum_{i=1}^{r} \sum_{j=0}^{l_i} m_j^i$. *Also assume that solutions of the characteristic equation*

$$p_m(\lambda - m, 0) = 0$$

are $\mu_i + n_j^i$ *with multiplicities* m_j^i *for* $i = 1, \dots, r$, $j = 0, \dots, l_i$.

Then there exist following m *formal solutions of* $P(x, \partial)u = 0$,

$$f(\mu_i + n_j^i; k)(x) = x^{\mu_i + n_j^i} \sum_{l=0}^{h_j^i + k} \binom{h_j^i + k}{l} c_{h_j^i + k - l}^{ij}(x) (\log x)^l,$$

where $c_k^{ij}(x) \in \mathbb{C}[[x^{-1}]]$, $c_0^{i0}(\infty) \ne 0$ *and* $h_j^i = \sum_{k=1}^{j} m_k^i$.

Proof. Suppose that $x^{\mu}\sum_{s=0}^{\infty} c_x x^{-s}$ $(c_0 \neq 0)$ is a formal solution of $P(x,\partial)u = \sum_{s=0}^{\infty} x^{\rho-s}P_s(\partial)u = 0$. Then we have the following equations,

$$\sum_{k=0}^{i} c_{i-k}p_k(\mu - i, 0) = 0, \tag{3.3}$$

for $i = 0, 1, \ldots$. The assumption implies $p_k(\lambda, 0)$ for $k < m$ are identically zero and $p_m(\lambda, 0)$ is nonzero polynomial of λ of degree m. Hence this is a direct consequence of the Frobenius method. □

Proposition 3.7. *Let us take $P(x,\partial) \in \mathcal{K}\langle\partial\rangle$ as in Definition 3.5. Suppose that $P_i^{(j)}(0) = 0$ for $i + j < m$ and $P_0^{(m)}(0) \neq 0$. We take $\mu_1, \ldots, \mu_r \in \mathbb{C}$ such that $\mu_i - \mu_j \notin \mathbb{Z}$ $(i \neq j)$. Then the followings are equivalent,*

1. *There exist m formal series,*

$$f_{ij}(x) = x^{-\mu_i - j} + x^{-\mu_i - m_i}h_{ij}(x)$$

where $h_{ij}(x) \in \mathbb{C}[[x^{-1}]]$ for $i = 1, \ldots, l$ and $j = 0, \ldots, m_i - 1$ and these are solutions of $P(x,\partial)u = 0$.

2. $$p_k(-\mu_i - j - k; 0) = 0$$

for

$$\begin{gathered} i = 1, \ldots, r, \\ j = 0, \ldots, m_i - 1, \\ k = m, m+1, \ldots, m + m_i - j - 1. \end{gathered}$$

Proof. First we assume that 1 is true. We consider the equation

$$P(x,\partial)x^{-\mu_i - j}(c_0 + \sum_{s=m_i-j}^{\infty} c_s x^{-s}) = 0$$

where $i \in \{1, \ldots, r\}$, $j \in \{0, \ldots, m_i - 1\}$ and $c_0 \neq 0$. Since $c_s = 0$ for $1 < s < m_i - j$, the equation (3.3) tells us that

$$c_0 p_k(-\mu_i - j - k; 0) = 0,$$

for $m \leq k < m_i - j + m$. If the condition 2 is false, $p_k(-\mu_i - j - k; 0) \neq 0$ for some k. This contradicts $c_0 \neq 0$.

Conversely we assume that 2 is true. Let us consider the equation $P(x,\partial)x^{-\mu_i - j}(c_0 + \sum_{s=m_i-j}^{\infty} c_s x^{-s}) = 0$. Then the equation (3.3) implies

$$c_0 p_k(-\mu_i - j - k; 0) = 0,$$

for $n \le k < m_i - j + n$ and

$$\sum_{l=0}^{k} c_{m_i-j+l} p_{m+(k-l)}(-\mu_i - m_i - k - m, 0) \\ + c_0 p_{m_i-j+m+k}(-\mu_i - m_i - k - m, 0) = 0$$

for $k = 0, 1, \ldots$. By the assumption 2, we can choose $c_0 \neq 0$. Then if c_0 is determined, other coefficients c_s for $s = m_i - j, m_i - j + 1, \ldots$ are determined inductively. We can put $c_0 = 1$. Then we can choose $x^{-\mu_i - j}(1 + \sum_{s=m_i-j}^{\infty} c_s x^{-s})$ as a formal solution of $P(x, \partial)f(x) = 0$ for all $i \in \{1, \ldots, r\}$, $j \in \{0, \ldots, m_i - 1\}$. □

Definition 3.8 (Semi-simple characteristic exponents)**.** *Let us take a differential operator*

$$P(x, \partial) = \sum_{s=0}^{\infty} x^{\rho - s} P_s(\partial)$$

where $P_s(x) \in \mathbb{C}[x]$ and $\deg P_s \le n$ for $s = 0, 1, 2, \ldots$. Suppose that $P_i^{(j)}(0) = 0$ for $i + j < m$ and $P_0^{(m)}(0) \neq 0$. Also suppose that there exit $\mu_1, \ldots, \mu_r \in \mathbb{C}$ such that $\mu_i - \mu_j \notin \mathbb{Z}$ $(i \neq j)$ and these satisfy

$$p_k(-\mu_i - j - k, 0) = 0$$

for

$$i = 1, \ldots, r, \\ j = 0, \ldots, m_i - 1, \\ k = m, m+1, \ldots, m + m_i - j - 1.$$

Here $m = \sum_{i=1}^{r} m_i$. Then we say that $P(x, \partial)$ has semi-simple characteristic exponents

$$\{\mu_1, \mu_1 + 1, \ldots, \mu_1 + m_1 - 1, \ldots, \mu_r, \mu_r + 1, \ldots, \mu_r + m_r - 1\},$$

at $x = \infty$. By using the notation $[\mu]_m = \{\mu, \mu + 1, \ldots \mu_{m-1}\}$, we write

$$\{[\mu_1]_{m_1}, \ldots, [\mu_r]_{m_r}\} \\ = \{\mu_1, \mu_1 + 1, \ldots, \mu_1 + m_1 - 1, \ldots, \mu_r, \mu_r + 1, \ldots, \mu_r + m_r - 1\}$$

shortly.

Let us recall that

$$e^{-p(x)} \partial e^{p(x)} = \partial + p'(x).$$

For a $P(x,\partial) \in \mathcal{K}\langle\partial\rangle$, the differential equation $P(x,\partial)u = 0$ has a formal solution

$$e^{p(x)}x^{-\nu}\sum_{i=0} c_s x^{-s}$$

if and only if the differential equation $P(x, \partial + p'(x))$ has a formal solution

$$x^{-\nu}\sum_{i=0} c_s x^{-s}.$$

Definition 3.9 ($e^{p(x)}$-twisted semi-simple characteristic exponents)**.** *For $P(x,\partial) \in \mathcal{K}\langle\partial\rangle$ and $p(x) \in \mathbb{C}[x]$, we say that the differential equation $P(x,\partial)u = 0$ has $e^{p(x)}$-twisted semi-simple exponents*

$$\{[\mu_1]_{m_1}, \ldots, [\mu_l]_{m_l}\}$$

at $x = \infty$ where $\mu_i \in \mathbb{C}$, $m_i \in \mathbb{N}$ and $m = \sum_{i=1}^{l} m_i$, if the differential equation $P(x, \partial + p'(x))$ has the same semi-simple exponents at $x = \infty$.

Proposition 3.10. *Suppose that $P(x,\partial) \in W[x]$ has $e^{p(x)}$-twisted semi-simple exponents*

$$\{[\mu_1]_{m_1}, \ldots, [\mu_l]_{m_l}\}$$

at $x = \infty$ where $\mu_i \in \mathbb{C}$, $m_i \in \mathbb{N}$ and $m = \sum_{i=1}^{l} m_i$.

1. *For $\alpha, \nu \in \mathbb{C}$, the differential equation $\mathrm{Ad}((x-\alpha)^{\nu})P(x,\partial)u(x) = 0$ has $e^{p(x)}$-twisted semi-simple exponents*

   $$\{[\mu_1 - \nu]_{m_1}, \ldots, [\mu_l - \nu]_{m_l}\}$$

   at $x = \infty$.

2. *For $q(x) \in \mathbb{C}$, the differential equation $\mathrm{Ade}(q(x))P(x,\partial)$ has $e^{p(x)+q(x)}$-twisted semi-simple exponents*

   $$\{[\mu_1]_{m_1}, \ldots, [\mu_l]_{m_l}\}$$

   at $x = \infty$.

Proof. If we recall that $(x-\alpha)^{\nu}$ can be written as $(x-\alpha)^{\nu} = x^{-\nu}\sum_{i=0}^{\infty} c_i x^{-i}$, the first assertion follows from the same argument as Proposition 3.14. The second assertion easily follows from

$$\mathrm{Ade}(q(x))P(x,\partial) = \exp(q(x))P(x,\partial)\exp(-q(x)).$$

$\square$

3.2 A review of regular singularity

Let us take a differential operator $P(x,\partial) \in W[x]$. If $P(x,\partial)$ has a regular singular point at $x = c \in \mathbb{C}$, then we can write

$$P(x,\partial) = \sum_{i=0}^{n} (x-c)^{n-i} a_i(x) \partial^{n-i}$$

where $a_0(c) \neq 0$. We consider a polynomial of ν

$$f_c(x,\nu) = \sum_{j=0}^{n} \frac{a_j(x)}{a_0(x)} \nu(\nu-1)\cdots(\nu-(n-j)+1).$$

Since $x = c$ is a regular singular point of $P(x,\partial)$, $f_c(x,\nu)$ is holomorphic at $x = c$. Hence we have the Taylor expansion

$$f_c(x,\nu) = \sum_{k=0}^{\infty} f_k^c(\nu)(x-c)^k, \tag{3.4}$$

where $f_k^c(\nu)$ are polynomials of ρ. Then a power series $g(\nu, x) = (x-c)^\nu \sum_{k=0}^{\infty} d_k (x-c)^k$ satisfies $P(x,\partial) g(\nu, x) = 0$ if and only if equations

$$\sum_{k=0}^{l} c_{l-k} f_k^c(\nu + (l-k)) = 0 \tag{3.5}$$

are satisfied for $l = 0, 1, \ldots$. We call the equation $f_0^c(\rho) = 0$ the characteristic equation at regular singular point $x = c$.

Then we have a similar result to Proposition 3.7.

Proposition 3.11 (Oshima [11]). *Let us take $P(x,\partial) \in W[x]$ which has a regular singular point $x = c$. We write*

$$P(x,\partial) = \sum_{i=0}^{n} (x-c)^{n-i} a_i(x) \partial^{n-i}$$

where $a_i(x) \in \mathbb{C}[x]$ and $a_0(c) \neq 0$. Then we can define polynomials $f_k^c(\nu)$ for $k = 0, 1, 2, \ldots$ as (3.4). Then the followings are equivalent.

1. *There exist $\mu_1, \ldots, \mu_l \in \mathbb{C}$ such that $\mu_i - \mu_j \notin \mathbb{Z}$ if $i \neq j$. The following n functions are solutions of $P(x,\partial)u = 0$,*

   $$g_{ij}(x) = (x-c)^{\mu_i + j} + (x-c)^{\mu_i + m_i} h_{ij}(x-c)$$

   for $i = 1, \ldots, l$ and $j = 0, \ldots, m_i - 1$. Here $h_{ij}(x) \in \mathbb{C}[[x]]$ and $n = \sum_{i=1}^{l} m_i$.

2. *There exist $\mu_1, \ldots, \mu_l \in \mathbb{C}$ such that $\mu_i - \mu_j \notin \mathbb{Z}$ if $i \neq j$. For these μ_i, we have*
$$f_k(\mu_i - j) = 0$$
for $i = 1, \ldots, l$, $j = 0, \ldots, m_i - 1$ and $k = 0, \ldots, m_i - j - 1$.

Proposition 3.12 (Oshima [11])**.** *Let us take $P(x, \partial) \in W[x]$ which has a regular singular point $x = c$. We write*
$$P(x, \partial) = \sum_{i=0}^{n} (x - c)^{n-i} a_i(x) \partial^{n-i}$$
where $a_i(x) \in \mathbb{C}[x]$ and $a_0(c) \neq 0$. Then the followings are equivalent.

1. *There exist m functions,*
$$g_i(x) = (x - c)^i + (x - c)^m h_i(x - c)$$
for $i = 0, \ldots, m - 1$ and these are solutions of $P(x, \partial)u = 0$. Here $h_i(x) \in \mathbb{C}[[x]]$.

2. *There exist $Q(x, \partial) \in W[x]$ such that*
$$P(x, \partial) = (x - c)^m Q(x, \partial).$$

Proof. Although the proof of this proposition can be found in [11], we prove this for the completeness. Suppose that 2 is true. We notice that $Q(x, \partial)(x - c)^i$ are holomorphic at $x = c$ for any $i \in \mathbb{Z}_{\geq 0}$. Hence if we write $Q(x, \partial)(x - c)^i = h_i(x) \in \mathbb{C}[[x]]$, then we have
$$P(x, \partial)(x - c)^i = (x - c)^m Q(x, \partial)(x - c)^i = (x - c)^m h_i(x).$$
Conversely, if 1 is true, there exist $h_i(x) \in \mathbb{C}[[x]]$ such that
$$P(x, \partial)(x - c)^i = \sum_{j=0}^{n} (x - c)^{n-j} a_j(x) \partial^{n-j} (x - c)^i = (x - c)^m h_i(x),$$
for $i = 0, \ldots, m - 1$. If $i = 0$, we have
$$a_0(x) = (x - c)((x - c)^{m-1} h_0(x) - \sum_{j=1}^{n} (x - c)^{n-i-1} a_j(x)).$$
Hence $P(x, \partial) = (x - c)Q_0(x, \partial)$ for a $Q(x, \partial) \in W[x]$. If $i = 1$, we have
$$(x - c)(a_1(x) + a_0(x)) = (x - c)^2((x - c)^{m-2} - \sum_{j=2}^{n} (x - c)^{n-j-2} a_j(x)).$$
Hence $P(x, \partial) = (x - c)^2 Q_1(x, \partial)$ for a $Q_1(x, \partial) \in W[x]$. We can iterate these for $i = 0, 1, \ldots, m - 1$. ∎

We can define semi-simple characteristic exponents at regular singular points as we define for formal solutions.

Definition 3.13. *Let us take $P(x, \partial) \in W[x]$ which has a regular singular point $x = c$. We write*

$$P(x, \partial) = \sum_{i=0}^{n} (x - c)^{n-i} a_i(x) \partial^{n-i}$$

where $a_i(x) \in \mathbb{C}[x]$ and $a_0(c) \neq 0$. Then we can define polynomials $f_k^c(\nu)$ for $k = 0, 1, 2, \dots$ as (3.4). If there exist $\mu_1, \dots, \mu_l \in \mathbb{C}$ such that $\mu_i - \mu_j \notin \mathbb{Z}$ $(i \neq j)$ and we have

$$f_k(\mu_i - j) = 0$$

for $i = 1, \dots, l$, $j = 0, \dots, m_i - 1$ and $k = 0, \dots, m_i - j - 1$, then we say that $P(x, \partial)$ has semi-simple characteristic exponents

$$\{[\mu_1]_{m_1}, \dots, [\mu_l]_{m_l}\}$$

at $x = c$.

Proposition 3.14. *Let $P(x, \partial) \in W[x]$ has a regular singular point $x = c$ and semi-simple exponents*

$$\{[\rho_1]_{m_1}, \dots, [\rho_l]_{m_l}\}.$$

1. *For $\nu \in \mathbb{C}$, the differential equation $\mathrm{Ad}((x - c)^{\nu})P(x, \partial)u = 0$ has semi-simple exponents*

   $$\{[\rho_1 + \nu]_{m_1}, \dots, [\rho_l + \nu]_{m_l}\}$$

   at $x = c$.

2. *If $\alpha \neq c$, then the addition at $x = \alpha$ does not change the set of exponents at $x = c$ of $P(x, \partial)$, that is, $\mathrm{Ad}((x-\alpha)^{\nu})P(x, \partial)$ has semi-simple exponents*

   $$\{[\rho_1]_{m_1}, \dots, [\rho_l]_{m_l}\}$$

   at $x = c$ as well.

3. *For $p(x) \in \mathbb{C}$, the set of exponents at $x = c$ of $P(x, \partial)$ are not changed by $\mathrm{Ade}(p(x))$, that is, $\mathrm{Ade}(p(x))P(x, \partial)$ has semi-simple exponents*

   $$\{[\rho_1]_{m_1}, \dots, [\rho_l]_{m_l}\}$$

   at $x = c$ as well.

Proof. If a function $u(x)$ is a solution of $P(x,\partial)u(x)=0$, then for $\alpha,\nu\in\mathbb{C}$, the function $(x-\alpha)^\nu u(x)$ satisfies that

$$\begin{aligned}&\mathrm{Ad}((x-\alpha)^\nu)P(x,\partial)(x-\alpha)^\nu u(x)\\ &\qquad=(x-\alpha)^\nu P(x,\partial)(x-\alpha)^{-\nu}(x-\alpha)^\nu u(x)\\ &\qquad\qquad=(x-\alpha)^\nu P(x,\partial)u(x)=0.\end{aligned}$$

Hence if $\alpha=c$ and $u(x)=(x-c)^\rho\sum_{i=0}^\infty c_ix^i$ is a solution of $P(x,\partial)u(x)=0$ around $x=c$, then $(x-c)^\nu u(x)=(x-c)^{\rho+\nu}\sum_{i=0}^\infty d_ix^i$ is a solution of $\mathrm{Ad}((x-c)^\nu)P(x,\partial)v(x)=0$.

On the other hand, if $\alpha\neq c$, the function $(x-\alpha)^\nu$ is holomorphic at $x=c$. Hence we can write the Taylor expansion $(x-\alpha)^\nu=\sum_{i=0}^\infty e_ix^i$. This implies that $(x-\alpha)^\nu u(x)=\sum_{i=0}^\infty e_ix^i(x-c)^\rho\sum_{j=0}^\infty c_jx^i=(x-c)^\rho\sum_{i=0}^\infty f_ix^i$. Therefore exponents does not change.

For the final assertion, we recall that

$$\mathrm{Ade}(p(x))P(x,\partial)=\exp(p(x))P(x,\partial)\exp\left(-p(x)\right),$$

and $\exp((p(x))$ is holomorphic on $\mathbb{C}$. By the same argument as above, the final assertion follows. ∎

4 Differential equations of irregular rank 2 at infinity

The twisted Euler transform turns $P(x,\partial)\in W[x,\xi]$ into the other $Q(x,\partial)\in W[x,\xi]$. The question is how this transformation changes local datum of these differential operators. We focus on the case of the rank of irregular singularity at most 2 and give explicit descriptions about the changes of characteristic exponents by twisted Euler transforms.

4.1 The rank of irregular singularity, the Newton polygon

Let us recall the notions, the rank of irregular singularity and the Newton polygon of a differential operator.

Definition 4.1 (The rank of irregular singularity at infinity)**.** *Let us consider a linear differential equation,*

$$[x^n\partial^n+a_1(x)x^{n-1}\partial^{n-1}+a_2(x)x^{n-2}\partial^{n-2}+\cdots+a_n(x)]f(x)=0. \qquad (4.1)$$

Here coefficients $a_i(x)$ are Laurent series,

$$a_i(x)=x^{m_i}\sum_{k=0}^\infty a^i_kx^{-k}\ (a^i_0\neq 0),$$

where $m_i \in \mathbb{Z}$ *for* $i = 1, \dots, n$.

The rank of irregular singularity at infinity of (*4.1*) *is the number defined by*

$$q = \max\{\frac{m_i}{i} \mid i = 1, \dots, n\}.$$

Let us take $P(x, \partial) = \sum a_i(x)\partial^i \in \mathcal{K}\langle\partial\rangle$. Every $a_i(x)\partial^i$ associate the point $(i, i - v(a_i))$ of $\mathbb{N} \times \mathbb{Z}$. Then we define the Newton polygon $N(P)$ of P to be the convex hull of the set

$$\bigcup_i \{(x, y) \in \mathbb{R}^2 \mid x \leq i,\, y \geq i - v(a_i)\}.$$

Let $\{s_i = (u_i, v_i)\}_{0 \leq i \leq p}$ be the set of vertices of this polygon such that $0 = u_0 < u_1 < \dots < u_p = n$ ($n = \operatorname{ord} P$). Slopes of the edge connecting s_i and s_{i-1} are

$$\lambda_i = \frac{v_i - v_{i-1}}{u_i - u_{i-1}}$$

for $i = 0, \dots, p-1$. Clearly we have $\lambda_1 < \lambda_2 < \dots < \lambda_p$. We define lengths L_i of segments $[s_{i-1}, s_i]$ by $L_i = u_i - u_{i-1}$. We note that λ_p corresponds to the irregular rank at infinity of P. We refer [10],[12] and [14] for further things about Newton polygons.

Remark 4.2. *Let us consider the Newton polygon of* $P(x, \partial)$ *whose vertices* $s_0, s_1, \dots, s_r$, *slopes* $\lambda_1 < \lambda_2 < \dots < \lambda_r$ *and lengths of segments are* $L_1, \dots, L_r$. *Then for* $1 \leq i \leq r$, *there exist* $q \in \mathbb{Z}_{>0}$ *and* L_i *linearly independent formal solutions of* $P(x, \partial)u = 0$,

$$f_k^{ij}(x) = e^{p^{ij}(x)} x^{\mu_k^{ij}} h_k^{ij}(x^{\frac{1}{q}})$$

for $q \in \mathbb{Z}_{>0}$ $h_k^{ij}(x) \in \mathbb{C}[[x^{-1}]][\log x^{-1}]$. *Here*

$$p^{ij}(x) = \sum_{l=0}^{r^{ij}} a^{ij}(l) x^{\frac{r^i - l}{q}}$$

and $\frac{r_i}{q} = \lambda_i$.

4.2 Fourier-Laplace transform

As is well-known, the Fourier-Laplace transform exchanges regular singular points on $\mathbb{C}$ and irregular singular point of rank 1 at infinity. Let us see the way how the Fourier-Laplace transform changes ranks of irregular singularities and characteristic exponents of a differential operators of irregular rank at most 2.

Proposition 4.3. *Let us take* $P(x, \partial) \in W[x]$ *of* $\deg P = N$ *and* $\mu_1, \dots, \mu_l \in \mathbb{C}$ *such that* $\mu_i \notin \mathbb{Z}$ *and* $\mu_i - \mu_j \notin \mathbb{Z}$ $(i \neq j)$. *Then the followings are equivalent.*

1. *The differential equation $P(x,\partial)u=0$ has semi-simple exponents*

$$\{[\mu_1]_{m_1},\ldots,[\mu_l]_{m_l}\}$$

at $x=\infty$. Here $m=\sum_{i=1}^{l} m_i$.

2. *For the Fourier-Laplace transform $P(-\partial,x)$ of $P(x,\partial)$ has regular singular point at $x=0$ and $P(-\partial,x)u=0$ has the semi-simple characteristic exponents,*

$$\{[0]_{N-m}[\mu_1-1]_{m_1},\ldots,[\mu_l-1]_{m_l}\}$$

at $x=0$.

Proof. Suppose that 1 is true. From the assumption $\mu_i\notin\mathbb{Z}$, we can see $N\geq m$. If $N<m$, we can write

$$P(x,\partial)=\sum_{s=0}^{N}x^{N-s}P_s(\partial)\partial^{m-s}=Q(x,\partial)\partial^{N-m}$$

for $Q(x,\partial)\in W[x]$. Then polynomials $\sum_{i=0}^{N-m-1}a_ix$ for $a_i\in\mathbb{C}$ satisfy $P(x,\partial)u=0$. It contradicts our assumption.

If we write $P(x,\partial)=\sum_{s=0}^{N}x^{N-s}P_s(\partial)\partial^{\max\{m-s,0\}}$, the Laplace transform

$$\begin{aligned}P(-\partial,x)&=\sum_{s=0}^{N}(-\partial)^{N-s}P_s(x)x^{\max\{m-s,0\}}\\&=\sum_{s=0}^{N}Q_s(x)x^{\max\{m-s,0\}}(-\partial)^{N-s}\end{aligned}$$

for $P_s(x),Q_s(x)\in\mathbb{C}[x]$ and $P_0(x)=Q_0(x)$. By the assumption $P_0(0)=Q_0(0)\neq 0$, it follows that $x=0$ is a regular singular point of $P(\partial,x)u=0$.

Let us take a power series $x^\mu\sum_{s=0}^{\infty}d_sx^s$. Then we can see that

$$\begin{aligned}&\mathcal{L}P(x,\partial)x^\mu\sum_{s=0}^{\infty}d_sx^s=\sum_{j=0}^{N}(-\partial)^{N-j}P_j(x)x^\mu\sum_{s=0}^{\infty}d_sx^s\\&=\sum_{s=m}^{\infty}x^{\mu-N+s}\sum_{k=m}^{s}d_{s-k}\sum_{l=0}^{k}(-\mu-s+l)(-\mu-s+l+1)\cdots(-\mu-s+N-1)\frac{P_l^{(k-l)}(0)}{(k-l)!}\\&=\sum_{s=m}^{\infty}x^{\mu-N+s}\sum_{k=m}^{s}d_{s-k}\sum_{l=0}^{k}(-\mu-s+l)_{(k-l)}(-\mu-s+k)_{(N-k)}\frac{P_l^{(k-l)}(0)}{(k-l)!}\\&\qquad=\sum_{s=m}^{\infty}x^{\mu-N+s}\sum_{k=m}^{s}(-\mu-s+k)_{(N-k)}d_{s-k}p_k(-\mu-1,0).\end{aligned}$$

Therefore if $\mathcal{L}P(x,\partial)x^{\mu}\sum_{s=0}^{\infty} d_s x^s = 0$, then it must be satisfied that

$$\sum_{k=m}^{s} d_{s-k}(-\mu - s + k)_{(N-k)} p_k(-\mu - 1 - s, 0)$$

for $s = m, m+1, \ldots$. By the assumption we have

$$p_k(-\mu_i - j - k, 0) = 0$$

for

$$\begin{gathered} i = 1, \ldots, l, \\ j = 0, \ldots, m_i - 1, \\ k = m, m+1, \ldots, m + m_i - j - 1. \end{gathered}$$

Also we have

$$(-i - s + k)_{(N-k)} = 0$$

for $i = 0, \ldots, (N-m) - 1$ and $k = m, m+1, \ldots, N - j - 1$. Hence the differential equation $\mathcal{L}P(x,\partial)u = 0$ has a regular singular point at $x = 0$ and semi-simple characteristic exponents

$$\{[0]_{N-m}, [\mu_1 - 1]_{m_1}, \ldots, [\mu_l - 1]_{m_l}\}.$$

The converse direction can be shown by the same way. □

The same thing can be shown for the Fourier-Laplace inverse transform.

Proposition 4.4. *Let us take $P(x,\partial) \in W[x]$ of $\deg P = N$ and $\mu_1, \ldots, \mu_l \in \mathbb{C}$ such that $\mu_i \notin \mathbb{Z}$ and $\mu_i - \mu_j \notin \mathbb{Z}$ $(i \neq j)$. Then the followings are equivalent.*

1. *The differential equation $P(x,\partial)u = 0$ has semi-simple exponents*

   $$\{[\mu_1]_{m_1}, \ldots, [\mu_l]_{m_l}\}$$

   at $x = \infty$. Here $m = \sum_{i=1}^{l} m_i$.

2. *For the Fourier-Laplace inverse transform $P(\partial, -x)$ of $P(x,\partial)$ has a regular singular point at $x = 0$ and $P(\partial, -x)u = 0$ has the semi-simple characteristic exponents,*

   $$\{[0]_{N-m}[\mu_1 - 1]_{m_1}, \ldots, [\mu_l - 1]_{m_l}\}$$

   at $x = 0$.

Proof. The condition 1 is equivalent to that $P(-x,-\partial)$ has semi-simple exponents

$$\{[\mu_1]_{m_1},\dots,[\mu_l]_{m_l}\}$$

at $x=\infty$. This is equivalent to that $P(\partial,-x)$ has semi-simple exponents

$$\{[0]_{N-m}[\mu_1-1]_{m_1},\dots,[\mu_l-1]_{m_l}\}$$

at $x=0$ by Proposition 4.3. □

Corollary 4.5. *Let us take $P(x,\partial)\in W[x]$ of $\deg P=N$ and $\mu_1,\dots,\mu_l\in\mathbb{C}$ such that $\mu_i\notin\mathbb{Z}$ and $\mu_i-\mu_j\notin\mathbb{Z}$ $(i\neq j)$. Then the followings are equivalent.*

1. *The differential equation $P(x,\partial)u=0$ has $e^{\alpha x}$-twisted semi-simple exponents*

   $$\{[\mu_1]_{m_1},\dots,[\mu_l]_{m_l}\}.$$

2. *For the Laplace transform $P(-\partial,x)$ of $P(x,\partial)$ has a regular singular point at $x=\alpha$ and semi-simple exponents*

   $$\{[0]_{N-m}[\mu_1-1]_{m_1},\dots,[\mu_l-1]_{m_l}\}$$

   at $x=\alpha$. Here $m=\sum_{i=1}^{l}m_i$.

Proof. The condition 1 is equivalent to that $P(x,\partial+\alpha)u=0$ has semi-simple exponents

$$\{[\mu_1]_{m_1},\dots,[\mu_l]_{m_l}\}$$

at $x=\infty$. Hence it is equivalent to that the Laplace transform $P(-\partial,x+\alpha)u=0$ has a regular singular point at $x=0$, i.e., $\mathcal{L}P(x,\partial)=P(-\partial,x)$ has a regular singular point at $x=\alpha$ and semi-simple exponents

$$\{[0]_{N-n}[\mu_1-1]_{m_1},\dots,[\mu_l-1]_{m_l}\}$$

at $x=\alpha$. □

For the inverse transform we can show the following as well.

Corollary 4.6. *Let us take $P(x,\partial)\in W[x]$ of $\deg P=N$ and $\mu_1,\dots,\mu_l\in\mathbb{C}$ such that $\mu_i\notin\mathbb{Z}$ and $\mu_i-\mu_j\notin\mathbb{Z}$ $(i\neq j)$. Then the followings are equivalent.*

1. *The differential equation $P(x,\partial)u=0$ has $e^{\alpha x}$-twisted semi-simple exponents*

   $$\{[\mu_1]_{m_1},\dots,[\mu_l]_{m_l}\}.$$

2. *For the Fourier-Laplace inverse transform $P(\partial,-x)$ of $P(x,\partial)$ has a regular singular point at $x=-\alpha$ and semi-simple exponents*

   $$\{[0]_{N-m}[\mu_1-1]_{m_1},\dots,[\mu_l-1]_{m_l}\}$$

   at $x=-\alpha$. Here $m=\sum_{i=1}^{l}m_i$.

Proof. We can show this by the same argument as Corollary 4.5. ☐

Corollary 4.7. *Let us take $P(x,\partial) \in W[x]$ of $\deg P = N$ and $\mu_1, \ldots, \mu_l \in \mathbb{C}$ such that $\mu_i \notin \mathbb{Z}$ and $\mu_i - \mu_j \notin \mathbb{Z}$ $(i \neq j)$. Then the followings are equivalent.*

1. *The differential equation $P(x,\partial)u = 0$ of has $e^{\frac{\alpha}{2}x^2+\beta x}$-twisted semi-simple exponents*
$$\{[\mu_1]_{m_1}, \ldots, [\mu_l]_{m_l}\}.$$
2. *The Laplace transform $P(-\partial, x)u = 0$ has $e^{-\frac{1}{2\alpha}x^2+\frac{\beta}{\alpha}x}$-twisted semi-simple exponents*
$$\{[\mu_1]_{m_1}, \ldots, [\mu_l]_{m_l}\}.$$

Proof. The condition 1 is equivalent to that $P(x, \partial + \alpha x + \beta)u = 0$ has semi-simple exponents
$$\{[\mu_1]_{m_1}, \ldots, [\mu_l]_{m_l}\}.$$
at $x = \infty$. On the other hand, the condition 2 is equivalent to that $P(-\partial + \frac{1}{\alpha}x - \frac{\beta}{\alpha}, x)u = 0$ has the same semi-simple exponents at infinity. If we put $x = \alpha y$, it is equivalent to say that $P(y - \frac{1}{\alpha}\partial_y - \frac{\beta}{\alpha}, \alpha y)v = 0$ has the same semi-simple exponents at infinity. Here $\partial_y = \frac{d}{dy}$. It is easy to see that
$$\mathcal{L} \circ \mathrm{Ade}((\frac{1}{2\alpha}y^2 - \frac{\beta}{\alpha}y)) \circ \mathcal{L}^{-1}P(y - \frac{1}{\alpha}\partial_y - \frac{\beta}{\alpha}, \alpha y) = P(y, \partial_y + \alpha y + \beta).$$
If we notice that for a solution u of $Q(x,\partial)u = 0$, $v = e^{p(x)}u$ is a solution of $\mathrm{Ade}(p(x))Q(x,\partial)v = 0$. Since $e^{p(x)}$ is holomorphic at $x = 0$, the multiplication of $e^{p(x)}$ does not change exponents at $x = 0$. Then the equivalence 1 and 2 follows from Proposition 4.3. ☐

Corollary 4.8. *Let us take $P(x,\partial) \in W[x]$ of $\deg P = N$ and $\mu_1, \ldots, \mu_l \in \mathbb{C}$ such that $\mu_i \notin \mathbb{Z}$ and $\mu_i - \mu_j \notin \mathbb{Z}$ $(i \neq j)$. Then the followings are equivalent.*

1. *The differential equation $P(x,\partial)u = 0$ has $e^{\frac{\alpha}{2}x^2+\beta x}$-twisted semi-simple exponents*
$$\{[\mu_1]_{m_1}, \ldots, [\mu_l]_{m_l}\}.$$
2. *The Fourier-Laplace inverse transform $P(\partial, -x)u = 0$ has $e^{-\frac{1}{2\alpha}x^2-\frac{\beta}{\alpha}x}$-twisted semi-simple exponents*
$$\{[\mu_1]_{m_1}, \ldots, [\mu_l]_{m_l}\}.$$

Proof. The condition 2 is equivalent to that $P(\partial - \frac{1}{\alpha}x - \frac{\beta}{\alpha}, -x)$ has the above semi-simple exponents at $x = \infty$. If we put $y = -\frac{1}{\alpha}x$, this is equivalent to $P(-\frac{1}{\alpha}\partial_y + y - \frac{\beta}{\alpha}, \alpha y)$ has the same exponents at $y = \infty$. Also we have
$$\mathcal{L}^{-1} \circ \mathrm{Ade}(\frac{1}{2\alpha}y^2 + \frac{\beta}{\alpha}) \circ \mathcal{L}P(-\frac{1}{\alpha}\partial_y + y - \frac{\beta}{\alpha}, \alpha y) = P(y, \partial_y + \alpha y + \beta).$$
As in Corollary 4.7, we can show this corollary. ☐

Proposition 4.9. *Let us take $P(x,\partial) \in W[x]$. We assume that $P(x,\partial)$ can be written by*

$$\mathrm{R}P(x,\partial) = x^N \prod_{i=0}^{r} (\partial - \alpha_i)^{m_i} + \sum_{j=1}^{N-1} x^{N-j} \prod_{i}^{r} (\partial - \alpha_i)^{\max\{m_i - j, 0\}} P_j(\partial)$$

for $P_i(x) \in \mathbb{C}[x]$ and $P(x,\partial)$ has $e^{\alpha_i x}$-twisted semi-simple exponents

$$\{[\mu_1^i]_{m_1^i}, \ldots, [\mu_{l_i}^i]_{m_{l_i}^i}\}$$

where $m_i = \sum_{j=1}^{l_i} m_j^i$. Moreover we assume that $\mu_j^i \notin \mathbb{Z}$ for $i = 1, \ldots, r$ and $j = 1, \ldots, l_i$ and $\mu_j^i - \mu_k^i \notin \mathbb{Z}$ $(k \neq j)$.

Then we have the followings.

1. *We have*
$$E(*;0)P(x,\partial) \sim P(x,\partial)$$
for $i = 1, \ldots, r$.

2. *For fixed $i \in \{1, \ldots, r\}$ and $\mu \in \mathbb{C}$ such that $\mu \notin \mathbb{Z}$ and $\mu_j^i - \mu \notin \mathbb{Z} \backslash \{1\}$ for all $j = 1, \ldots l_i$, we have*
$$E(\alpha_i; -\mu) E(\alpha_i; \mu) P(x,\partial) \sim P(x,\partial)$$
for $i = 1, \ldots, r$.

Proof. First we show 1. We have

$$\begin{aligned} E(\alpha_i; 0)P(x,\partial) &= \mathcal{L}\mathrm{R}\mathcal{L}^{-1}\mathrm{R}P(x,\partial) \\ &= \mathcal{L}\mathrm{R}(\prod_{i=1}^{r}(-x-\alpha_i)^{m_i}\partial^N \\ &+ \sum_{j=1}^{N}\prod_{i=1}^{r}(-x-\alpha_i)^{\max\{m_i-j,0\}} a_j(x)\partial^{N-j}). \end{aligned}$$

By the assumption and Corollary 4.6, every characteristic exponent at the regular singular point $x = -\alpha_i$ of $\mathcal{L}^{-1}\mathrm{R}P(x,\partial)$ is not integers. Hence by Proposition 3.12, $\mathrm{R}\mathcal{L}^{-1}\mathrm{R}P(x,\partial) = \mathcal{L}^{-1}\mathrm{R}P(x,\partial)$. Hence $E(*;0)P(x,\partial) = \mathcal{L}\mathrm{R}\mathcal{L}^{-1}\mathrm{R}P(x,\partial) \sim P(x,\partial)$.

Let us show 2. Fix an $i \in \{1, \ldots r\}$. If $\mu_j^i - \mu \neq 1$ for all $j = 1, \ldots, l_i$, $E(\alpha_i, \mu)P(x,\partial)$ has $e^{\alpha_i x}$-twisted semi-simple exponents

$$\{[-\mu+1]_{N-m_i}[\mu_1^i - \mu]_{m_1^i}, \ldots, [\mu_{l_i}^i - \mu]_{m_{l_i}^i}\}$$

and $e^{\alpha_{i'} x}$-twisted semi-simple exponents

$$\{[\mu_1^{i'}]_{m_1^{i'}}, \ldots, [\mu_{l_{i'}}^{i'}]_{m_{l_{i'}}^{i'}}\}$$

for the other i' by corollaries from 4.5 to 4.8. On the other hand, if there exist $j \in \{1, \ldots, l_i\}$ such that $\mu^i_j - 1 = u$, then $E(\alpha_i, \mu)P(x, \partial)$ has $e^{\alpha_i x}$-twisted semi-simple exponents

$$\{[\mu^i_1 - \mu]_{m^i_1}, \ldots, [\mu^i_{j-1} - \mu]_{m^i_{j-1}}, [-\mu+1]_{N-m_i}, [\mu^i_{j+1} - \mu]_{m^i_{j+1}}, \ldots, [\mu^i_{l_i} - \mu]_{m^i_{l_i}}\}$$

and $e^{\alpha_{i'} x}$-twisted semi-simple exponents

$$\{[\mu^{i'}_1]_{m^{i'}_1}, \ldots, [\mu^{i'}_{l_{i'}}]_{m^{i'}_{l_{i'}}}\}$$

for the other i' by corollaries from 4.5 to 4.8 In both cases, characteristic exponents are not integers and moreover difference of them are not integers. Hence we can see

$$E(*, 0)E(\alpha_i, \mu)P \sim E(\alpha_i, \mu)P$$

by the same argument as 1. Hence we have 2 as well as the proof of Proposition 2.12.

□

4.3 Twisted Euler transform

Definition 4.10 (Normal at infinity). *Let us take $P(x, \partial) \in W[x, \xi]$ which has an irregular singular point at $x = \infty$ of rank 2. The order of $P(x, \partial)$ is n.*

We say that $P(x, \partial)$ is normal at the irregular singular point $x = \infty$, if the followings are satisfied. There exist $\alpha_i, \beta^i_j \in \mathbb{C}$ for $i = 1, \ldots, r$ and $j = 1, \ldots, l_i$.

1. *For every $i = 1, \ldots, r$, the Newton polygon of $P(x, \partial + \alpha_i x)$ has only three vertices $s^i_0 = (u^i_0 = 0, v^i_0), s^i_1 = (u^i_1, v^i_1)$ and $s^i_3 = (u^i_3 = n, v^i_3)$. Corresponding slopes are 1 and 2. Length of the segment $[s^i_1, s^i_0]$ is n_i. Here $n = \sum_{i=1}^r n_i$.*

2. *For every (i, j), $i = 1, \ldots, r$ and $j = 1, \ldots, l_i$, the Newton polygon of $P(x, \partial + \alpha_i x + \beta^i_j)$ has only four vertices $s^{ij}_0 = (u^{ij}_0 = 0, v^{ij}_0), s^{ij}_1 = (u^{ij}_1, v^{ij}_1), s^{ij}_2 = (u^{ij}_2, v^{ij}_2), s^{ij}_3 = (u^{ij}_3 = n, v^{ij}_3)$. Corresponding slopes are 0, 1 and 2. Lengths of segments $[s^{ij}_1, s^{ij}_0]$, $[s^{ij}_2, s^{ij}_1]$ and $[s^{ij}_3, s^{ij}_2]$ are n^i_j, $n_i - n^i_j$ and $n - n_i$ respectively. Here $n_i = \sum_{j=1}^{l_i} n^i_j$.*

Remark 4.11. *By Proposition 3.4, Proposition 3.6 and Remark 4.2, $P(x, \partial) \in W[x, \xi]$ is normal at infinity if and only if there are n^i_j-dimensional space of $e^{\frac{\alpha_i}{2}x^2 + \beta^i_j x}$-twisted formal solutions. Since $n = \sum_{i=1}^r \sum_{j=1}^{l_i} n^i_j$ coincides with the order of $P(x, \partial)$, these are all of formal solutions of $P(x, \partial)u = 0$ at $x = \infty$.*

Let us take $P(x,\partial) \in W[x,\xi]$ which has regular singular points at $c_1,\dots,c_p \in \mathbb{C}$, the irregular singular point of rank 2 at $x=\infty$ and no other singular points. Then we can write

$$P(x,\partial)=\sum_{i=0}^{n}\prod_{j=1}^{p}(x-c_j)^{n-i}a_i(x)\partial^{n-i}, \tag{4.2}$$

where $a_i(x) \in \mathbb{C}[x,\xi]$ for $i=1,\dots,n$ and $a_0(x)=1$. Since the rank of irregularity is 2, degrees of $a_i(x)$ have upper bound,

$$\deg a_i(x) \le (p+1)i.$$

We call (4.2) the standard form of $P(x,\partial)$.

Definition 4.12 (Table of local datum)**.** *For sufficiently large numbers K and K', we take $\xi=\{\mu^i_j \mid 1\le i,j\le K\}\cup\{\nu^{ij}_k \mid 1\le i,j,k\le K'\}$ as the indeterminants of $\mathbb{C}(\xi)$. Let us take $P(x,\partial)\in W[x,\xi]$ which has regular singular points at $c_1,\dots,c_p\in\mathbb{C}$, the irregular singular point of rank 2 at $x=\infty$ and no other singular points. Moreover $P(x,\partial)$ is normal at infinity.*

Let us assume that the differential equation $P(x,\partial)u=0$ has the following local solutions.

- *Around each regular singular point $x=c_i$, it has semi-simple exponents*

  $$\{[0]_{m^i_0}[\mu^i_1]_{m^i_1},\dots,[\mu^i_{s_i}]_{m^i_{s_i}}\}$$

  where $m^i_j\in\mathbb{Z}_{>0}$ for $i=1,\dots,s_i$ and $m^i_0\in\mathbb{Z}_{\ge 0}$ which satisfy $\sum_{j=0}^{s_i}m^i_j = n=\operatorname{ord}P$ for $i=1,\dots,p$. For the simplicity, we put

  $$\mu^i=(\mu^i_1,\dots,\mu^i_{s_i}), \qquad m^i=(m^i_1,\dots,m^i_{s_i}),$$

  and write

  $$[\mu^i;m^i]=\{[\mu^i_1]_{m^i_1},\dots,[\mu^i_{s_i}]_{m^i_{s_i}}\},$$

  shortly.

- *Around irregular singular point $x=\infty$, it has $e^{\frac{\alpha_i}{2}x^2+\beta_j x}$-twisted semi-simple exponents*

  $$\{[\nu^{ij}_1]_{n^{ij}_1},\dots,[\nu^{ij}_{t_{ij}}]_{n^{ij}_{t_{ij}}}\}$$

  for $i=1,\dots,r$, $j=1,\dots,l_i$. Here $n^{ij}_k\in\mathbb{Z}_{>0}$ and $\sum_{i=1}^{r}\sum_{j=1}^{l_i}\sum_{k=1}^{t_{ij}}n^{ij}_k = n=\operatorname{ord}P(x,\partial)$. We put $n^i_j=\sum_{k=1}^{t_{ij}}n^{ij}_k$ and $n_i=\sum_{j=1}^{l_i}n^i_j$. For the simplicity, we put

  $$\nu^{ij}=(\nu^{ij}_1,\dots,\nu^{ij}_{t_{ij}}), \qquad n^{ij}=(n^{ij}_1,\dots,n^{ij}_{t_{ij}}),$$

  and write

  $$[\nu^{ij};n^{ij}]=\{[\nu^{ij}_1]_{n^{ij}_1},\dots,[\nu^{ij}_{t_{ij}}]_{n^{ij}_{t_{ij}}}\}.$$

Then we write the following table of local exponents of $P(x, \partial)$,

		α_1		α_2		$\cdots$	α_r	
c_1	$[\mu^1; m^1]$	β^1_1	$[\nu^{11}; n^{11}]$	β^2_1	$[\nu^{21}; n^{21}]$	$\cdots$	β^r_1	$[\nu^{r1}; n^{r1}]$
c_2	$[\mu^2; m^2]$	β^1_2	$[\nu^{12}; n^{12}]$	β^2_2	$[\nu^{22}; n^{22}]$	$\cdots$	β^r_2	$[\nu^{r2}; n^{r2}]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
c_p	$[\mu^p; m^p]$	$\beta^1_{l_1}$	$[\nu^{1l_1}; n^{1l_1}]$	$\beta^2_{l_2}$	$[\nu^{2l_2}; n^{2l_2}]$		$\beta^r_{l_r}$	$[\nu^{rl_r}; n^{rl_r}]$

.

We call this table the table of local datum of $P(x, \partial)$.

In particular, if $P(x, \partial) = \partial + \alpha x + \beta$ for some $\alpha, \beta \in \mathbb{C}$, we say $P(x, \partial)$ has the trivial table of local datum.

Remark 4.13. *In Definition 4.12, the indeterminants $\xi = \{\mu^i_j \mid 1 \leq i, j \leq K\} \cup \{\nu^{ij}_k \mid 1 \leq i, j, k \leq K'\}$ have only one linear relation which comes from Fuchs relation (see [1] and [2] for example).*

Theorem 4.14. *Let us take $P(x, \partial) \in W[x, \xi]$ as in Definition 4.12. We assume that $P(x, \partial)$ has the following nontrivial table of local datum,*

		α_1		α_2		$\cdots$	α_r	
c_1	$[\mu^1; m^1]$	β^1_1	$[\nu^{11}; n^{11}]$	β^2_1	$[\nu^{21}; n^{21}]$	$\cdots$	β^r_1	$[\nu^{r1}; n^{r1}]$
c_2	$[\mu^2; m^2]$	β^1_2	$[\nu^{12}; n^{12}]$	β^2_2	$[\nu^{22}; n^{22}]$	$\cdots$	β^r_2	$[\nu^{r2}; n^{r2}]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
c_p	$[\mu^p; m^p]$	$\beta^1_{l_1}$	$[\nu^{1l_1}; n^{1l_1}]$	$\beta^2_{l_2}$	$[\nu^{2l_2}; n^{2l_2}]$		$\beta^r_{l_r}$	$[\nu^{rl_r}; n^{rl_r}]$

.

Changing the order of $\alpha_1, \ldots, \alpha_r$ and applying $Ade(\frac{\alpha_1}{2}x^2)$, We can assume that $\alpha_1 = 0$. Then $\mathcal{L} \circ \mathrm{R}\, P(x, \partial)$ has

$$\mathrm{ord}\, \mathcal{L} \circ \mathrm{R} P(x, \partial) = \sum_{i=1}^{p} \sum_{j=1}^{s_i} m^i_j + (n - n_1),$$

and the table of local datum,

		0		$-\frac{1}{\alpha_2}$		$\cdots$	$-\frac{1}{\alpha_r}$	
β^1_1	$[\tilde{\nu}^{11}; n^{11}]$	$-c_1$	$[\tilde{\mu}^1; m^1]$	$\frac{\beta^2_1}{\alpha_2}$	$[\nu^{21}; n^{21}]$	$\cdots$	$\frac{\beta^r_1}{\alpha_r}$	$[\nu^{r1}; n^{r1}]$
β^1_2	$[\tilde{\nu}^{12}; n^{12}]$	$-c_2$	$[\tilde{\mu}^2; m^2]$	$\frac{\beta^2_2}{\alpha_2}$	$[\nu^{22}; n^{22}]$	$\cdots$	$\frac{\beta^r_2}{\alpha_r}$	$[\nu^{r2}; n^{r2}]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$\beta^1_{l_1}$	$[\tilde{\nu}^{1l_1}; n^{1l_1}]$	$-c_p$	$[\tilde{\mu}^p; m^p]$	$\frac{\beta^2_{l_2}}{\alpha_2}$	$[\nu^{2l_2}; n^{2l_2}]$		$\frac{\beta^r_{l_r}}{\alpha_r}$	$[\nu^{rl_r}; n^{rl_r}]$

.

Here

$$\tilde{\mu}^i = (\mu^i_1 + 1, \mu^i_2 + 1, \ldots, \mu^i_{s_i} + 1), \quad \widetilde{\nu^{1j}} = (\nu^{1j}_1 - 1, \nu^{1j}_2 - 1, \ldots, \nu^{1j}_{t_{1j}} - 1),$$

for $i = 1, \ldots, p$ and $j = 1, \ldots, l_1$.

Proof. By Proposition 3.11, the standard form of $P(x,\partial)$ can be divided by $\phi(x) = \prod_{i=1}^{p}(x-c_i)^{m_i}$, i.e., there exist $Q(x,\partial) \in W[x,\xi]$ such that $P(x,\partial) = \phi(x)Q(x,\partial)$ and moreover $\mathrm{R}P(x,\partial) = Q(x,\partial)$. Since we assume that $\alpha_1 = 0$, we can see $\deg P(x,\partial) = (p+1)(n-n_1)$. Hence $\deg Q(x,\partial) = \deg P(x,\partial) - \sum_{i=1}^{p} m_i = (p+1)(n-n_1) - \sum_{i=1}^{p} m_i = \sum_{i=1}^{p}(n-m_0^i) + (n-n_1) = \sum_{i=1}^{p}\sum_{j=1}^{s_i} m_j^i + (n-n_1)$.

Then this theorem is obtained by Corollary 4.5 and Corollary 4.7. □

Remark 4.15. *We can show the same thing as the above theorem for the Fourier-Laplace inverse transform by Corollary 4.6 and Corollary 4.8.*

Definition 4.16 (The rigidity index)**.** *Let us take $P(x,\partial) \in W[x,\xi]$ as in Definition 4.12. Then we define the number*

$$\begin{aligned}&\operatorname{idx} P = \\ &-((p+1)n^2 - \sum_{i=1}^{r} n_i^2 - \sum_{i=1}^{r}\sum_{j=1}^{l_i}(n_j^i)^2 - \sum_{i=1}^{p}\sum_{j=0}^{s_i}(m_j^i)^2 - \sum_{i=1}^{q}\sum_{j=1}^{l_i}\sum_{k=1}^{t_{ij}}(n_k^{ij})^2),\end{aligned}$$

and call this the rigidity index of P.

Remark 4.17. *The standard form of $P(x,\partial)$ is*

$$P(x,\partial) = \sum_{i=0}^{n}\prod_{j=1}^{p}(x-c_j)^{n-i}a_i(x)\partial^{n-i},$$

where

$$\deg a_i(x) \le (p+1)i.$$

Hence $P(x,\partial)$ has

$$\sum_{i=1}^{n}((p+1)i+1)$$

coefficients in $\overline{\mathbb{C}(\xi)}$. The informations about the sets of exponents at regular singular points $x = c_1,\ldots,c_p$ require

$$\sum_{i=1}^{p}\sum_{j=0}^{s_i}\frac{m_j^i(m_j^i+1)}{2}$$

linear equations by Lemma 3.11. On the other hand, $P(x,\partial)$ has n_i-dimensional formal solutions with $e^{\frac{\alpha_i}{2}x^2+\beta_j^i x}$-twisted semi-simple exponents for $i = 1,\ldots,r$ and $j = 1,\ldots,l_i$. Hence the standard form of $P(x,\partial+\alpha_i x)$ is

$$P(x,\partial+\alpha_i x) = \sum_{k=0}^{n}\prod_{l=1}^{p}(x-c_l)^{n-k}b_k^i(x)\partial^{n-k},$$

where

$$\begin{cases}\deg b^i_k(x) \le (p+1)k & \textit{for } 1 \le k \le (n-n_i)\\ \deg b^i_{(n-n_i)+m}(x) \le pm + (p+1)(n-n_i) & \textit{for } 1 \le m \le n_i\end{cases}.$$

Hence this requires

$$\sum_{k=1}^{n}(\deg a_k(x) - \deg b^i_k(x)) = \frac{n_i(n_i+1)}{2}$$

linear equations of $\overline{\mathbb{C}(\xi)}$*. Moreover* $P(x, \partial + \alpha_i x + \beta^i_j)$ *has the standard form*

$$P(x, \partial + \alpha_i x + \beta^i_j) = \sum_{k=0}^{n}\prod_{l=1}^{p}(x-c_l)^{n-k} c^{ij}_k(x)\partial^{n-k},$$

where

$$\begin{cases}\deg c^{ij}_k(x) \le (p+1)k & \textit{for } 1 \le k \le (n-n_i)\\ \deg c^{ij}_{(n-n_i)+l}(x) \le pl + (p+1)(n-n_i) & \textit{for } 1 \le l \le (n_i - n^i_j)\\ \deg c^{ij}_{(n-(n_i-n^i_j))+m} \le (p-1)m + p(n_i - n^i_j) + (p+1)(n-n_i) & \textit{for } 1 \le m \le n^i_j\end{cases}.$$

Hence this requires

$$\sum_{k=1}^{n}(\deg b^i_k(x) - \deg c^{ij}_k(x)) = \frac{n^i_j(n^i_j+1)}{2}$$

linear equations of $\overline{\mathbb{C}(\xi)}$*. Finally, informations about the sets of exponents require*

$$\sum_{k=1}^{t_{ij}}\frac{n^{ij}_k(n^{ij}_k+1)}{2}$$

linear equations of $\overline{\mathbb{C}(\xi)}$ *by Proposition 3.7. There is one more linear equation from Fuchs relation. Hence there are at most the following parameters in* $P(x,\partial)$*,*

$$\sum_{i=1}^{n}((p+1)i+1) - \sum_{i=1}^{p}\sum_{j=0}^{s_i}\frac{m^i_j(m^i_j+1)}{2} - \sum_{i=1}^{r}\frac{n_i(n_i+1)}{2}$$
$$- \sum_{i=1}^{r}\sum_{j=1}^{l_i}\frac{n^i_j(n^i_j+1)}{2} - \sum_{i=1}^{p}\sum_{j=1}^{l_i}\sum_{k=1}^{t_{ij}}\frac{n^{ij}_k(n^{ij}_k+1)}{2} + 1 = 1 - \frac{1}{2}\mathrm{idx}\, P.$$

Theorem 4.18. *Let us take* $P(x,\partial) \in W[x,\xi]$ *as in Definition 4.12 with the table of local datum,*

		α_1		α_2		$\cdots$	α_r	
c_1	$[\mu^1; m^1]$	β^1_1	$[\nu^{11}; n^{11}]$	β^2_1	$[\nu^{21}; n^{21}]$	$\cdots$	β^r_1	$[\nu^{r1}; n^{r1}]$
c_2	$[\mu^2; m^2]$	β^1_2	$[\nu^{12}; n^{12}]$	β^2_2	$[\nu^{22}; n^{22}]$	$\cdots$	β^r_2	$[\nu^{r2}; n^{r2}]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
c_p	$[\mu^p; m^p]$	$\beta^1_{l_1}$	$[\nu^{1l_1}; n^{1l_1}]$	$\beta^2_{l_2}$	$[\nu^{2l_2}; n^{2l_2}]$		$\beta^r_{l_r}$	$[\nu^{rl_r}; n^{rl_r}]$

.

1. *For* $f(\xi) \in \mathbb{C}(\xi)$ *and* $i = 1, \ldots, p$*, the table of local datum of* $\mathrm{RAd}((x - c_i)^{f(\xi)})P(x, \partial)$ *is*

		α_1		$\cdots$	α_r	
c_1	$[\mu^1; m^1]$	β^1_1	$[\nu^{11} - f(\xi); n^{11}]$	$\cdots$	β^r_1	$[\nu^{r1} - f(\xi); n^{r1}]$
c_2	$[\mu^2; m^2]$	β^1_2	$[\nu^{12} - f(\xi); n^{12}]$	$\cdots$	β^r_2	$[\nu^{r2} - f(\xi); n^{r2}]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
c_i	$[\mu^i + f(\xi); m^i]$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
c_p	$[\mu^p; m^p]$	$\beta^1_{l_1}$	$[\nu^{1l_1} - f(\xi); n^{1l_1}]$		$\beta^r_{l_r}$	$[\nu^{rl_r} - f(\xi); n^{rl_r}]$

.

where $\mu^i + f(\xi) = (\mu^i_1 + f(\xi), \ldots, \mu^i_{s_i} + f(\xi))$ *and* $\nu^{ij} - f(\xi) = (\nu^{ij}_1 - f(\xi), \ldots, \nu^{ij}_{t_{ij}} - f(\xi))$.

2. *For* $i = 1, \ldots, r$*,* $j = 1, \ldots, l_i$ *and* $k = 1, \ldots, t_{ij}$*, the table of local datum of* $E(\frac{\alpha_i}{2}x^2 + \beta^i_j x; \beta^i_j, \nu^{ij}_k - 1)P(x, \partial)$ *is*

		α_1		α_2		$\cdots$	α_r	
c_1	$[\tilde{\mu}^1; m^1]$	β^1_1	$[\tilde{\nu}^{11}; \tilde{n}^{11}]$	β^2_1	$[\tilde{\nu}^{21}; \tilde{n}^{21}]$	$\cdots$	β^r_1	$[\tilde{\nu}^{r1}; \tilde{n}^{r1}]$
c_2	$[\tilde{\mu}^2; m^2]$	β^1_2	$[\tilde{\nu}^{12}; \tilde{n}^{12}]$	β^2_2	$[\tilde{\nu}^{22}; \tilde{n}^{22}]$	$\cdots$	β^r_2	$[\tilde{\nu}^{r2}; \tilde{n}^{r2}]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
c_p	$[\tilde{\mu}^p; m^p]$	$\beta^1_{l_1}$	$[\tilde{\nu}^{1l_1}; \tilde{n}^{1l_1}]$	$\beta^2_{l_2}$	$[\tilde{\nu}^{2l_2}; \tilde{n}^{2l_2}]$		$\beta^r_{l_r}$	$[\tilde{\nu}^{rl_r}; \tilde{n}^{rl_r}]$

,

where

$$\tilde{\nu}^{xy} =$$

$$\begin{cases} (\nu^{ij}_1 - (\nu^{ij}_k - 1), \ldots, \nu^{ij}_{k-1} - (\nu^{ij}_k - 1), -\nu^{ij}_k, \nu^{ij}_{k+1} - (\nu^{ij}_k - 1), \ldots) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } (x, y) = (i, j), \\ \nu^{iy} \ \text{if } x = i \text{ and } y \neq j, \\ (\nu^{xy}_1 + \nu^{ij}_k - 1, \ldots, \nu^{xy}_{t_{xy}} + \nu^{ij}_k - 1) \ \text{otherwise}, \end{cases}$$

$$\tilde{\mu}^x = (\mu^x_1 + \nu^{ij}_k - 1, \ldots, \mu^x_{s_x} + \nu^{ij}_k - 1),$$

$$\tilde{n}^{xy} = \begin{cases} (n^{ij}_1, \ldots, n^{ij}_{k-1}, N_i - n^i_j, n^{ij}_{k+1}, \ldots) & \text{if } (x, y) = (i, j), \\ n^{xy} & \text{otherwise}, \end{cases}$$

for $N_i = \sum_{k=1}^{p}\sum_{l=1}^{s_k} m_l^k + (n - n_i)$. *The order of* $E(\frac{\alpha_i}{2}x^2 + \beta_j^i x; \beta_j^i, \nu_k^{ij} - 1)P(x, \partial)$ *is* $n - n_k^{ij} + N_i - n_j^i$.

Proof. The first assertion is the direct consequence of Proposition 3.14 and Proposition 3.10. The second assertion follows from Theorem 4.14 and the first assertion. □

Remark 4.19. *In the second assertion of Theorem 4.18, we see that the order of* $E(\frac{\alpha_i}{2}x^2 + \beta_j^i x; \beta_j^i, \nu_k^{ij} - 1)P(x, \partial)$ *is* $(n - n_k^{ij}) + (N - n_j^i)$. *By the same argument as the proof of Proposition 4.3, we can see that* $N_i - n_j^i \geq 0$ *for all* i *and* j. *Hence* $E(\frac{\alpha_i}{2}x^2 + \beta_j^i x; \beta_j^i, \nu_k^{ij} - 1)P(x, \partial)$ *is well-defined, i.e.,* $(n - n_k^{ij}) + (N_i - n_j^i) > 0$, *if and only if* $n - n_k^{ij} > 0$ *or* $N_i - n_j^i > 0$. *On the contrary, if we assume* $n = n_k^{ij}$ *and* $N_i = n_j^i$, *we can see the following.*

Lemma 4.20. *Let us take* $P(x, \partial) \in W[x, \xi]$ *whose table of local datum is*

			α
c_1	$[\mu^1; m^1]$	β	$[(0); (n)]$
c_2	$[\mu^2; m^2]$		
$\vdots$	$\vdots$		
c_p	$[\mu^p; m^p]$		

.

And we assume $n = \sum_{i=1}^{p}\sum_{j=1}^{s_i} m_j^i$. *Then*

$$P(x, \partial) \sim (\partial - \alpha x - \beta)^n.$$

Proof. We can see that $x = \infty$ is a regular singular point of $P(x, \partial + \alpha x + \beta)$. Hence we can write

$$P(x, \partial + \alpha x + \beta) = \sum_{i=0}^{n} x^{n-i} P_i(\partial),$$

where $\deg P_i(x) \leq n$ for $i = 0, \ldots, n$ and $P_i^{(j)}(0) = 0$ for $i + j < n$. We consider polynomials

$$p_k(\lambda, x) = \sum_{i=0}^{k} (\lambda + i + 1)_{k-i} \frac{P_i^{(k-i)}(x)}{(k-i)!}.$$

The characteristic exponents at $x = \infty$ implies that

$$p_{n+j}(-i - j, 0) = 0$$

for $i = 0, 1, \ldots, n-1$ and $j = 0, \ldots, n-i-1$. This implies that $P_i^{(j)}(0) = 0$ for $i = 1, \ldots, n$ and $j = 0, 1, \ldots n-1$. Hence we have

$$P(x, \partial + \alpha x + \beta) \sim \partial^n,$$

i.e.,

$$P(x,\partial) \sim (\partial - \alpha x - \beta)^n.$$

$\square$

Corollary 4.21. *Let us take $P(x,\partial) \in W[x,\xi]$ as in Definition 4.12 with the table of local datum,*

		α	
c_1	$[\mu^1; m^1]$	β	$[(\nu); (n)]$
c_2	$[\mu^2; m^2]$		
$\vdots$	$\vdots$		
c_p	$[\mu^p; m^p]$		

.

And we assume that $n = \sum_{i=1}^r \sum_{j=1}^{s_i} m_j^i$. Then we have

$$E(\frac{\alpha}{2}x^2; \beta, \nu)P \sim (\partial - \alpha x - \beta)^n$$

Proof. By Theorem 4.18, we can see that $E(\frac{\alpha}{2}x^2; \beta, \nu)P$ has the table of local datum

		α	
c_1	$[\mu^1; m^1]$	β	$[(0); (n)]$
c_2	$[\mu^2; m^2]$		
$\vdots$	$\vdots$		
c_p	$[\mu^p; m^p]$		

.

Therefore by Lemma 4.20, we have

$$E(\frac{\alpha}{2}x^2; \beta, \nu)P \sim (\partial - \alpha x - \beta)^n.$$

$\square$

Proposition 4.22. *Let us take $P(x,\partial) \in W[x,\xi]$ as in Definition 4.12 and assume that $P(x,\partial)$ has the nontrivial table of local datum*

		α_1		α_2		$\cdots$	α_r	
c_1	$[\mu^1; m^1]$	β_1^1	$[\nu^{11}; n^{11}]$	β_1^2	$[\nu^{21}; n^{21}]$	$\cdots$	β_1^r	$[\nu^{r1}; n^{r1}]$
c_2	$[\mu^2; m^2]$	β_2^1	$[\nu^{12}; n^{12}]$	β_2^2	$[\nu^{22}; n^{22}]$	$\cdots$	β_2^r	$[\nu^{r2}; n^{r2}]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
c_p	$[\mu^p; m^p]$	$\beta_{l_1}^1$	$[\nu^{1l_1}; n^{1l_1}]$	$\beta_{l_2}^2$	$[\nu^{2l_2}; n^{2l_2}]$		$\beta_{l_r}^r$	$[\nu^{rl_r}; n^{rl_r}]$

.

Then we have the followings.

1. *We have*
$$E(\frac{\alpha_i}{2}x^2; *, 0)P \sim P.$$
for $i = 1, \ldots, r$ and for any complex number $*$.

2. Let us take $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, l_i\}$ and fix them. If $\nu_k^{ij} - f(\xi) \notin \mathbb{Z} \backslash \{1\}$ for all $k = 1, \dots, t_{ij}$, then we have

$$E(\frac{\alpha_i}{2}x^2; \beta_j^i; -f(\xi))E(\frac{\alpha_i}{2}x^2; \beta_j^i; f(\xi))P \sim P.$$

Proof. By the assumption, we can write

$$\mathrm{R}P(x, \partial + \alpha_i x) = x^N \prod_{j=1}^{l_i} (\partial - \beta_j^i)^{n_j^i} + \sum_{k=1}^{N} x^{N-k} \prod_{j=1}^{l_i} (\partial - \beta_j^i)^{\max\{n_j^i - k\}} P_k(\partial)$$

for $P_k(x) \in \mathbb{C}[x]$ and $N = n - n_i + \sum_{k=1}^{p} \sum_{l=1}^{s_k} m_l^k$. Hence we can apply Proposition 4.9. $\square$

Although the twisted Euler transform in Corollary 4.21 does not satisfy the assumption of 2 in Proposition 4.22, we can show the following.

Proposition 4.23. *Let us take $P(x, \partial)$ as in Definition 4.12 with the table of local datum,*

$$\begin{array}{|cc||cc|}
\hline
 & & & \alpha \\
\hline
c_1 & [\mu^1; m^1] & \beta & [(\nu); (n)] \\
c_2 & [\mu^2; m^2] & & \\
\vdots & \vdots & & \\
c_p & [\mu^p; m^p] & & \\
\hline
\end{array}.$$

Then we have

$$E(\frac{\alpha}{2}x^2; \beta, -\nu)E(\frac{\alpha}{2}x^2; \beta, \nu)P \sim P.$$

Proof. Without loss of the generality, we can assume $\alpha = 0$. The table of local datum of $\mathrm{RAd}((x+\beta)^{-\nu})\mathcal{L}\mathrm{R}P$ is

$$\begin{array}{|cc||cc|}
\hline
 & & & 0 \\
\hline
-\beta & [(0, -1); (N, n)] & c_1 & [\mu'^1; m^1] \\
 & & c_2 & [\mu'^2; m^1] \\
 & & \vdots & \vdots \\
 & & c_p & [\mu'^p; m^p] \\
\hline
\end{array}.$$

Here $N = \sum_{i=1}^{r} \sum_{j=1}^{s_i} m_j^i - n$ and

$$\mu'^i = (\mu_1^i + \nu + 1, \dots, \mu_{s_i}^i + \nu + 1)$$

for $i = 1, \dots, r$. Hence by the same argument as in Proposition 4.9, we can show that

$$(\mathcal{L}^{-1}\mathrm{R}\mathcal{L})\mathrm{RAd}((x+\beta)^{-\nu})\mathcal{L}^{-1}P \sim \mathrm{RAd}((x+\beta)^{-\nu})\mathcal{L}^{-1}P.$$

Hence we have

$$\begin{aligned}E(\beta,-\nu)E(\beta,\nu)P &= \mathcal{L}\mathrm{RAd}((x+\beta)^{\nu})\mathcal{L}^{-1}\mathrm{R}\mathcal{L}\mathrm{RAd}((x+\beta)^{-\nu})\mathcal{L}^{-1}\mathrm{R}P\\ &= \mathcal{L}\mathrm{RAd}((x+\beta)^{\nu})\mathrm{RAd}((x+\beta)^{-\nu})\mathcal{L}^{-1}P\\ &= \mathcal{L}\mathrm{R}\mathcal{L}^{-1}P = E(*,0)P\\ &\sim P.\end{aligned}$$

□

5 Kac-Moody root system

P. Boalch found a correspondence between quiver varieties and moduli spaces of meromorphic connections on vector bundles over the Riemann sphere of the forms

$$(\frac{A}{z^3}+\frac{B}{z^2}+\frac{C}{z})dz.$$

He studied the existence of these meromorphic connections through the theory of representations of quiver varieties which is first studied by W. Crawley-Boevey in [4].

In this section, as an analogue of this result of Boalch, we attach a differential operator considered in Definition 4.12 to a Kac-Moody Lie algebra and an element of the root lattice of this algebra. And we show the equivalence between twisted Euler transforms and additions on differential equations and the action of Weyl group on the corresponding element of the root lattice.

Let us take $P(x,\partial)\in W[x,\xi]$ with the table of local datum

$$\begin{array}{|cc||cc|cc|c|cc|}\hline & & \alpha_1 & & \alpha_2 & & \cdots & \alpha_r & \\ \hline c_1 & [\mu^1;m^1] & \beta^1_1 & [\nu^{11};n^{11}] & \beta^2_1 & [\nu^{21};n^{21}] & \cdots & \beta^r_1 & [\nu^{r1};n^{r1}]\\ c_2 & [\mu^2;m^2] & \beta^1_2 & [\nu^{12};n^{12}] & \beta^2_2 & [\nu^{22};n^{22}] & \cdots & \beta^r_2 & [\nu^{r2};n^{r2}]\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots\\ c_p & [\mu^p;m^p] & \beta^1_{l_1} & [\nu^{1l_1};n^{1l_1}] & \beta^2_{l_2} & [\nu^{2l_2};n^{2l_2}] & & \beta^r_{l_r} & [\nu^{rl_r};n^{rl_r}]\\ \hline\end{array}.$$

We fix this $P(x,\partial)$ through this section.

Let $\mathfrak{h}$ be the complex vector space with the basis

$$\Pi=\{v^{ij}_k \mid i=0,\dots,r, j=1,\dots,l_i, k=1,\dots,t^{ij}\}.$$

Here we put $l_0=p$ and $t^{0j}=s_j$. We define the non-degenerate symmetric bilinear form on $\mathfrak{h}$ as follows,

$$(v^{ij}_k,v^{lm}_n)=\begin{cases}2 & \text{if } (i,j,k)=(l,m,n)\\ -1 & \text{if } (i,j)=(l,m) \text{ and } |k-n|=1\\ -1 & \text{if } (k,n)=(1,1) \text{ and } i\neq j\\ 0 & \text{otherwise}\end{cases}.$$

If for $s = (i_s, j_s, k_s) \in I = \{(i,j,k) \mid i = 0, \ldots, r, j = 1, \ldots, l_i, k = 1, \ldots, t^{ij}\}$, we write $v_s = v_{k_s}^{i_s j_s}$. We can define the generalized Cartan matrix,

$$A = \left(\frac{2(v_s, v_t)}{(v_s, v_s)} \right)_{s \in I, t \in I}.$$

Let $\mathfrak{g}(A)$ be the Kac-Moody Lie algebra with the above generalized Cartan matrix A. According to the usual terminology, we call Π the root basis, elements from Π are called simple roots and $\mathbb{Z}$-lattice generated by Π, i.e.

$$Q = \sum_{(i,j,k) \in I} \mathbb{Z} v_k^{ij}$$

is called the root lattice. Also we define the positive root lattice

$$Q^+ = \sum_{(i,j,k) \in I} \mathbb{Z}_{\geq 0} v_k^{ij}.$$

The height of an element of the root lattice $\alpha = \sum_{i,j,k \in I} x_k^{ij} v_k^{ij}$ is defined by

$$\mathrm{ht}(\alpha) = \sum_{(i,j,k) \in I} x_k^{ij}.$$

Also the support of α is defined by

$$\operatorname{supp} \alpha = \{v_k^{ij} \in \Pi \mid x_k^{ij} \neq 0\}.$$

We say that the subset $L \subset \Pi$ is connected if the decomposition $L_1 \cup L_2 = L$ with $L_1 \neq \emptyset$ and $L_2 \neq \emptyset$ always implies the existence of $v_i \in L_i$ satisfying $(v_1, v_2) \neq 0$.

We have the following root space decomposition of $\mathfrak{g}(A)$ with respect to $\mathfrak{h}$,

$$\mathfrak{g}(A) = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g}(A) \mid [H, X] = (\alpha, H)X \text{ for all } H \in \mathfrak{h}\}$ is the root space attached to α. The root space is $\Delta = \{\alpha \in Q \mid \mathfrak{g}_\alpha \neq \{0\}\}$ and we call elements of Δ roots.

The reflections on $\mathfrak{h}$ with respect to simple roots v_k^{ij}, so-called simple reflections, are defined by

$$r_k^{ij} : \mathfrak{h} \ni H \longmapsto r_k^{ij}(H) = H - \frac{2(H, v_k^{ij})}{(v_k^{ij}, v_k^{ij})} v_k^{ij} = H - (H, v_k^{ij}) v_k^{ij}.$$

The Weyl group W is the group generated by all simple reflections.

A root $\alpha \in \Delta$ is called real root if there exists $w \in W$ such that $w(\alpha) \in \Pi$. We denote the set of real roots by Δ^{re}. A root which is not real root is called

imaginary root. We denote the set of all imaginary roots by Δ^{im}. Hence there is a decomposition of the set of roots, $\Delta = \Delta^{re} \cup \Delta^{im}$. If the Cartan matrix is symmetrizable, we can see that

$$\Delta^{re} = \{\alpha \in \Delta \mid (\alpha, \alpha) > 0\}, \qquad \Delta^{im} = \{\alpha \in \Delta \mid (\alpha, \alpha) \leq 0\}.$$

In our case the Cartan matrix A is symmetric. Hence moreover we have

$$\Delta^{re} = \{\alpha \in \Delta \mid (\alpha, \alpha) = 2\}.$$

For fundamental things about Kac-Moody Lie algebra, we refer the standard text book [7].

For the above $P \in W[x, \xi]$, we can define the element $\alpha_P \in Q^+$ associated with $P(x, \partial)$ as follows. Let us put

$$\tilde{n}_k^{ij} = \begin{cases} \sum_{l=k}^{s_j} m_l^j & \text{if } i = 0 \\ \sum_{l=k}^{t_{ij}} n_l^{ij} & \text{if } i = 1, \ldots, r \end{cases}.$$

Then $\alpha_P \in Q^+$ is defined by

$$\alpha_P = \sum_{(i,j,k) \in I} \tilde{n}_k^{ij} v_k^{ij}.$$

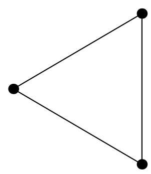
Remark 5.1. *This correspondence between $P \in W[x, \xi]$ and $\alpha_P \in Q^+$ is not unique. Indeed, for ν_k^{ij} and n_k^{ij}, permutations with respect to the index $k = 1, \ldots, t^{ij}$ do not change local solutions of P. For μ_j^i and m_j^i, permutations with respect to the index $j = 1, \ldots, s_i$ do not change local solutions of P as well.*

Example 5.2. *If $P(x, \partial)$ has the table of local datum,*

		α_1		α_2	
c_1	$[(\mu_1^1); (1)]$	β_1	$[(\nu_1^1); (1)]$	β_2	$[(\nu_1^2); (1)]$

,

the corresponding Kac-Moody Lie algebra has the following dynkin diagram,



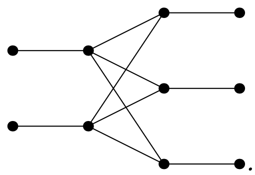
.

If $P(x, \partial)$ has the table of local datum,

		α	
c_1	$[(\mu_1^1, \mu_2^1); (1, 1)]$	β_1	$[(\nu_1^1, \nu_1^2); (1, 1)]$
c_2	$[(\mu_1^2, \mu_1^2); (1, 1)]$	β_2	$[(\nu_1^2, \nu_2^2); (1, 1)]$
		β_3	$[(\nu_1^3, \nu_1^3); (1, 1)]$

,

the corresponding Kac-Moody Lie algebra has the following dynkin diagram,



Theorem 5.3. *We retain the above notations.*

1. *For* $P \in W[x,\xi]$ *and* $\alpha_P \in Q^+$ *defined as above, we have*
$$\operatorname{idx} P = (\alpha_P, \alpha_P).$$

2. *Let us assume* $n - n_k^{ij} > 0$ *or* $N_i - n_j^i > 0$ *where* $N_i = \sum_{k=1}^{l_0} \sum_{l=1}^{t_{0k}} m_l^k + (n - n^i)$. *If we put* $Q = E(\frac{\alpha_i}{2}x^2 + \beta_j^i x; \beta_j^i, \nu_1^{ij} - 1)P(x, \partial)$, *then we have*
$$\alpha_Q = r_1^{ij}(\alpha_P).$$

3. *If we put* $Q = \mathrm{RAd}((x - c_i)^{-\mu_1^i})P$, *then we have*
$$\alpha_Q = r_1^{0i}(\alpha_P).$$

4. *For* $k \geq 2$, *reflections* $r_k^{ij}(\alpha_P)$ *correspond following permutations of the table of local datum of* P,
$$(\mu_z^y, m_z^y) \mapsto \begin{cases} (\mu_{z+1}^y, m_{z+1}^y) & \textit{if } i = 0, j = y \textit{ and } k = z \\ (\mu_{z+1}^y, m_{z-1}^y) & \textit{if } i = 0, j = y \textit{ and } k = z - 1 \\ (\mu_z^y, m_z^y) & \textit{otherwise} \end{cases},$$
$$(\nu_z^{xy}, n_z^{xy}) \mapsto \begin{cases} (\nu_{z+1}^{xy}, n_{z+1}^{xy}) & \textit{if } (x,y,z) = (i,j,k) \\ (\nu_{z-1}^{xy}, n_{z-1}^{xy}) & \textit{if } (x,y,z-1) = (i,j,k) \\ (\nu_z^{xy}, n_z^{xy}) & \textit{otherwise} \end{cases}.$$
Here μ_z^y *and* ν_z^{xy} *are exponents of local solutions of* P *and* m_z^y *and* n_z^{xy} *are multiplicities of them respectively.*

This theorem immediately follows from the following lemma.

Lemma 5.4. *Let us take* $\alpha \in \mathfrak{h}$ *such that*
$$\alpha = \sum_{(i,j,k) \in I} c_k^{ij} v_k^{ij}$$

for $c_k^{ij} \in \mathbb{C}$. The we have the following equations,

$$\begin{aligned}(\alpha,\alpha) =& -(l_0+1)(\sum_{i=1}^{r}\sum_{j=1}^{l_i} c_1^{ij})^2 + \sum_{i=1}^{r}(\sum_{j=1}^{l_i} c_1^{ij})^2 + \sum_{i=1}^{r}\sum_{j=1}^{l_i}(c_1^{ij})^2 \\ &+ \sum_{i=1}^{l_0}(\sum_{j=1}^{r}\sum_{k=1}^{l_j} c_1^{jk} - c_1^{0i}) + \sum_{i=0}^{r}\sum_{j=1}^{l_i}\sum_{k=1}^{t_{ij}}(c_k^{ij} - c_{k+1}^{ij})^2,\end{aligned}$$

$$(\alpha, v_{k_0}^{i_0j_0}) = \begin{cases} -(\sum_{i=1}^{r}\sum_{j=1}^{l_i}\sum_{k=1}^{t_{ij}} c_k^{ij} - \sum_{i=1}^{l_{i_0}}\sum_{j=1}^{t_{i_0 i}} c_k^{i_0 j}) + c_1^{i_0j_0} + (c_1^{i_0j_0} - c_2^{i_0j_0}) & \text{if } k_0 = 1 \\ (c_{k_0}^{i_0j_0} - c_{k_0-1}^{i_0j_0}) + (c_{k_0}^{i_0j_0} - c_{k_0+1}^{i_0j_0}) & \text{if } k_0 \geq 2 \end{cases}.$$

Proof. Direct computation. □

We consider a subset of Q^+,

$$V = \{nv_1^{i'j'} + \sum_{j=1}^{l_0}\sum_{k=1}^{n} m_k^j v_k^{0j} \mid {}^{1\leq i'\leq r, 1\leq j'\leq l_{i'}}_{n, m_k^j \in \mathbb{Z}_{>0} \text{ such that } n \geq m_1^j > m_2^j > \cdots}\}$$

Proposition 5.5. *If $\alpha_P \notin W(V)$, the set of all Weyl group orbits of elements of V, then the Weyl group orbit of α_P, $W(\alpha_P)$, is contained in Q^+.*

Proof. If $\alpha_P \notin V$, a twisted Euler transform of $P(x,\partial)$ corresponds to a simple reflection of α_P by Theorem 5.3. Hence we can find $Q(x,\partial) \in W[x,\xi]$ which corresponds to $\alpha_Q = r_1^{ij}(\alpha_P)$ by taking the twisted Euler transform of $P(x,\partial)$. Moreover if $\alpha_Q \notin V$, we can find $Q'(x,\partial) \in W[x,\xi]$ which corresponds to $\alpha_{Q'} = r_1^{i'j'}(\alpha_Q) = r_1^{i'j'} r_1^{ij}(\alpha_P)$ by the twisted Euler transform. Hence if $\alpha_P \notin W(V)$, we can iterate these. Also we can use same argument for other simple reflections. Then we have the proposition. □

Corollary 5.6. *If $\alpha_P \notin W(V)$, then $\alpha_P \in \Delta^{im}$.*

Proof. First we assume that $(\alpha_P, \alpha_P) > 0$. Let β be an element of minimal height among $W(\alpha_P) \cap Q^+$. Since the Weyl group action does not change inner product, we have $(\beta,\beta) > 0$. Hence we have $(\beta, v_k^{ij}) > 0$ for some $(i,j,j) \in I$. If $\beta \neq v_k^{ij}$, then $r_k^{ij}(\beta) \in Q^+$ and $\mathrm{ht}(r_k^{ij}(\beta)) < \mathrm{ht}(\beta)$, a contradiction with the choice of β. Hence $\beta = v_k^{ij}$. Since $\alpha_P \notin W(V)$, we can find $Q_\beta \in W[x,\xi]$ such that $\alpha_{Q_\beta} = \beta$. This implies that if β is the simple root, then $\beta = v_1^{ij}$ for some i and j. However $v_1^{ij} \in V$. This contradicts our assumption. Hence if $\alpha_P \notin W(V)$, we have $(\alpha_P, \alpha_P) \leq 0$.

Now we assume that $(\alpha_P, \alpha_P) \leq 0$. As above, we choose an element $\beta \in W(\alpha_P)$ of minimal height. Then $(\beta, v_k^{ij}) \leq 0$ for all $(i,j,k) \in I$. Since

β corresponds to some $Q_\beta \in W[x,\xi]$, this implies that $\operatorname{supp}\beta$ is connected. Hence

$$\beta \in K = \{\alpha \in Q^+ \backslash \{0\} \mid {\scriptstyle (\alpha, v_k^{ij}) \leq 0 \text{ for all } (i,j,k) \in I \atop \text{and } \operatorname{supp}\alpha \text{ is connected}}\}.$$

This implies that $\alpha_P \in W(K) = \Delta^{im} \cap Q^+$. ◻

Theorem 5.7. *Let us take $P(x,\partial) \in W[x,\xi]$ as in Definition 4.12. If $\operatorname{idx} P > 0$, then $P(x,\partial)$ can be reduced to*

$$(\partial - \alpha x - \beta)^n$$

for some $\alpha, \beta \in \mathbb{C}$ and $n \in \mathbb{Z}_{>0}$ by finite iterations of twisted Euler transforms and additions at regular singular points.

Proof. By Corollary 5.6, if $\operatorname{idx} P = (\alpha_P, \alpha_P) > 0$, then $\alpha_P \in W(V)$. Hence finite iterations of simple reflections α_P reduces to an element of V. This implies P reduces to a $Q \in W[x,\xi]$ with a table of local datum

		α	
c_1	$[\mu^1; m^1]$	β	$[(\nu); (n)]$
c_2	$[\mu^2; m^2]$		
$\vdots$	$\vdots$		
c_p	$[\mu^p; m^p]$		

,

by finite iterations of twisted Euler transforms and additions at regular singular points. Proposition 4.21 says that

$$E(\frac{\alpha}{2}x^2; \beta, \nu)Q \sim (\partial - \alpha x - \beta)^n.$$

Hence we have the theorem. ◻

6 Confluence

In this section, we show that differential operator $P(x,\partial)$ of $\operatorname{idx}P > 0$ can be obtained by the limit transition from a Fuchsian differential operator of the same rigidity index.

6.1 Fuchsian differential equations

Definition 6.1. *Let us take $P(x,\xi) \in W[x,\xi]$ as in Definition 4.12. If $P(x,\partial)$ has the following table of local datum,*

		0	
c_1	$[\mu^1; m^1]$	0	$[(\nu_1, \ldots, \nu_t); (n_1, \ldots, n_t)]$
c_2	$[\mu^2; m^2]$		
$\vdots$	$\vdots$		
c_p	$[\mu^p; m^p]$		

,

then we say that $P(x, \partial)$ is Fuchsian.

Proposition 6.2. *If $P(x, \partial)$ is Fuchsian with nontrivial table of local datum, then $E(0, f(\xi))P$ and $\mathrm{RAd}((x-c)^{g(\xi)})P$ are Fuchsian for any $c \in \mathbb{C}$ and $f(\xi), g(\xi) \in \mathbb{C}(\xi)$.*

Proof. This is a collorary of Theorem 4.18. $\square$

6.2 Versal additions

We define the operator called versal additions. These operators are introduced by Oshima in [11].

For $a_1, \ldots, a_n \in \mathbb{C}$, we define a function

$$h_n(a_1, \ldots, c_n; x) = -\int_0^x \frac{t^{n-1}\, dt}{\prod_{1 \le i \le n}(1 - a_i t)}.$$

Then it is not hard to see that

$$e^{\lambda_n h_n(a_1, \ldots, a_n; x)} = \prod_{k=1}^{n} (1 - a_k x)^{\frac{\lambda_n}{a_k \prod_{\substack{1 \le i \le n \\ i \ne k}} (a_k - a_i)}}.$$

Definition 6.3 (Versal addition)**.** *We put*

$$\mathrm{AdV}(a_1, \ldots, a_n; \lambda_1, \ldots, \lambda_n) = \prod_{k=1}^{n} \mathrm{Ad}\left((x - \frac{1}{a_k})^{\sum_{l=k}^{n} \frac{\lambda_l}{a_k \prod_{\substack{1 \le i \le l \\ i \ne k}} (a_k - a_i)}} \right).$$

Proposition 6.4. *For $P(x, \partial) \in W[x]$, we have*

$$\lim_{a_1 \to 0} \mathrm{AdV}(a_1; \lambda_1) P(x, \partial) = \mathrm{Ade}(-\lambda_1 x) P(x, \partial),$$
$$\lim_{\substack{a_1 \to 0 \\ a_2 \to 0}} \mathrm{AdV}(a_1, a_2; \lambda_1, \lambda_2) P(x, \partial) = \mathrm{Ade}(\lambda_1 x^2 + \lambda_2 x) P(x, \partial).$$

Proof. If we recall that

$$\mathrm{AdV}(a_1; \lambda_1) \colon \partial = \partial - \frac{\frac{\lambda_1}{a_1}}{(x - \frac{1}{a_1})} = \partial + \frac{\lambda_1}{1 - a_1 x},$$

then we can see that

$$\lim_{a_1 \to 0} AdV(a_1; \lambda_1)\partial = \partial + \lambda_1 = \mathrm{Ade}(-\lambda_1 x)\partial.$$

Also the equation

$$
\begin{aligned}
\mathrm{AdV}(a_1,a_2;\lambda_1,\lambda_2)\partial &= \partial - \frac{\frac{\lambda_1}{a_1}+\frac{\lambda_2}{a_1(a_1-a_2)}}{x-\frac{1}{a_1}} - \frac{\frac{\lambda_2}{a_2(a_2-a_1)}}{x-\frac{1}{a_2}} \\
&= \partial + \frac{\lambda_1+\frac{\lambda_2}{(a_1-a_2)}}{1-a_1x} + \frac{\frac{\lambda_2}{(a_2-a_1)}}{1-a_2x} \\
&= \partial + \frac{\lambda_1}{1-a_1x} + \frac{\lambda_2 x}{(1-a_1x)(1-a_2x)},
\end{aligned}
$$

implies that

$$
\lim_{\substack{a_1\to 0\\ a_2\to 0}} \mathrm{AdV}(a_1,a_2;\lambda_1,\lambda_2)\partial = \partial + \lambda_1 + \lambda_2 x = \mathrm{Ade}(-\frac{\lambda_2}{2}x^2 - \lambda_1 x)\partial.
$$

□

Theorem 6.5. *Take a $P(x,\partial)\in W[x,\xi]$ as in Definition 4.12. If $\operatorname{idx} P>0$, then $P(x,\partial)$ can be obtained by the limit transition of a Fuchsian $Q(x,\partial)\in W[x,\xi]$ of $\operatorname{idx} Q = \operatorname{idx} P$.*

Proof. By Theorem 5.7, $P(x,\partial)$ is obtained by finite iterations of twisted Euler transforms and additions from $(\partial-\alpha x-\beta)^n$. Here $\partial-\alpha x-\beta = \lim_{\substack{a_1\to 0\\ a_2\to 0}} \mathrm{AdV}(a_1,a_2;-2\alpha,-\beta)\partial$. As we see in Remark 2.11, twisted Euler transforms are compositions of Euler transforms, $\mathrm{Ade}(\alpha x)$ and $\mathrm{Ade}(\beta x^2+\gamma x)$ for some $\alpha,\beta,\gamma\in\mathbb{C}$. Hence twisted Euler transforms can be obtained by the limit transitions of compositions of additions and Euler transforms by Proposition 6.4.

Therefore $P(x,\partial)$ can be seen as the limit of a Fuchsian $Q(x,\xi)$ which is obtained by Euler transforms and additions from $\mathrm{AdV}(a_1,a_2;-2\alpha,-\beta)\partial$.

Finally we notice that twisted Euler transforms do not change rigidity indices because the action of Weyl group does not change the inner product. □

Appendix

A Differential equations with regular singularity at $x=\infty$ and Euler transform

We consider differential equations with regular singular point at $x=\infty$ and arbitrary singularities at any other points in $\mathbb{C}$. And then we give a necessary and sufficient condition to reduce the rank of differential equation by Euler transform.

Theorem A.1. *Let us take $P(x,\partial) \in W[x]$ which has regular singular point at $x=\infty$ and semi-simple exponents*

$$\{[\mu_1]_{n_1},\ldots,[\mu_l]_{n_l}\},$$

where $\sum_{i=1}^{l} n_i = n = \operatorname{ord} P$, $\mu_i \notin \mathbb{Z}$ and $\mu_i - \mu_j \notin \mathbb{Z}$ if $i \neq j$. Then we have

$$\operatorname{ord} E(0,\mu_i - 1)P(x,\partial) < \operatorname{ord} P$$

if and only if

$$\deg P - \operatorname{ord} P < n_i.$$

Proof. Since $x=\infty$ is the regular singular point of $P(x,\partial) \in W[x]$, we can write

$$\mathrm{R}P(x,\partial) = \sum_{i=0}^{N} x^{N-i} \partial^{\max\{n-i,0\}} P_i(\partial)$$

for $P_i(x) \in \mathbb{C}[x]$ of $\deg P_i \leq n$ for $i=0,\ldots,N$ and $P_0(x)=1$. Here $N=\deg P$ and $n=\operatorname{ord} P$. Hence we have

$$\mathcal{L}^{-1}\mathrm{R}P = \sum_{i=0}^{N} \partial^{N-i}(-x)^{\max\{n-i\}} P_i(-x)$$

and this has regular singular point at $x=0$ and no other singular points in $\mathbb{C}$. Also this has semi-simple exponents,

$$\{[0]_{N-n},[\mu_1-1]_{n_1},\ldots,[\mu_l-1]_{n_l}\}$$

at $x=0$ by Proposition laplace inverse transform of regular point. And then we can see that $\mathrm{RAd}(x^{-\mu_i+1})\mathcal{L}^{-1}\mathrm{R}P$ has semi-simple exponents,

$$\{[-\mu_i+1]_{N-n},[\mu_1-\mu_i]_{n_1},\ldots,[\mu_{i-1}-\mu_i]_{n_{i-1}},[0]_{n_i}[\mu_{i+1}-\mu_i]_{n_i},\ldots\}$$

by Proposition 3.14. Hence by Proposition 3.12, we have

$$\deg \mathrm{RAd}(x^{-\mu_i+1})\mathcal{L}^{-1}\mathrm{R}P = n - n_i + (N-n).$$

This means that

$$\begin{aligned}\operatorname{ord} \mathcal{L}\mathrm{RAd}(x^{-\mu_i+1})\mathcal{L}^{-1}\mathrm{R}P &= E(0,\mu_i-1)P \\ &= n - n_i + (N-n).\end{aligned}$$

Hence we have the theorem.

□

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